UNIVERSITY
OF MICHIGAN
FEB 2 1961
MATHEMATICS

Mathematical Reviews

Published monthly by The American Mathematical Society

TABLE OF CONTENTS

Volume 22, Number 1A January 1961 Reviews 1–239

Math
QA
1
.M4236
vol. 23
Pt. 1
Jan.-June
1961

MATHEMATICAL REVIEWS

Published by
The American Mathematical Society, 190 Hope St., Providence 6, R.I.

Sponsored by

The American Mathematical Society
The Mathematical Association of America
The Institute of Mathematical Statistics
The Edinburgh Mathematical Society
Société Mathématique de France
Dansk Matematisk Forening
Het Wiskundig Genootschap te Amsterdam
The London Mathematical Society
Polskie Towarzystwo Matematyczne
Unión Matemática Argentina
Indian Mathematical Society
Unione Matematica Italiana
The Society for Industrial and Applied Mathematics

Edited by
J. L. Doob W. S. Massey J. V. Wehausen
S. H. Gould, *Executive Editor*
Chandler Davis, W. Freiberger, Chia Kuei Tsao *and* J. A. Zilber, *Associate Editors*
H. A. Pogorzelski, *Copy Editor*

Editorial Office
Mathematical Reviews, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to individual members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Mathematical Reviews is published in 1961 in twelve monthly issues, each in two parts, A and B; and a single index issue covering both parts. Reviews and pages are numbered consecutively with respect to the issue order 1A, 1B, . . . , 12A, 12B. When the letter A or B is prefixed to a review number, it indicates in which part the review appears.

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) **4** **(82)** (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

Copyright © 1961 by the American Mathematical Society

Printed in Great Britain by William Clowes and Sons, Limited, London and Beccles

6 1
3208
8818
6 vols.

Mathematical Reviews

Vol. 22, No. 1A January, 1961 Reviews 1–239

GENERAL

See also 5.

1:

Graham, L. A. ★Ingenious mathematical problems and methods. Dover Publications, Inc., New York, 1960. vii+237 pp. Paperbound: $1.45.

One hundred mathematical puzzles of the "narrative" type, with solutions and discussions.

2:

Singh, Jagjit. ★Great ideas of modern mathematics: Their nature and use. Dover Publications, Inc., New York, 1960. viii+312 pp. Paperbound: $1.55.

Popularized exposition, including calculus, set theory, group theory, relativity, probability and logic.

3:

Sutton, O. G. ★Mathematics in action. Harper Torchbooks/The Science Library. Harper & Brothers, New York, 1960. xvi+236 pp. Paperbound: $1.45.

A reprinting of the second edition [G. Bell and Sons, Ltd., London, 1957], this fairly technical popularization, for the adept layman or beginning student, of the role of mathematics in physical theories includes detailed essays on ballistics and numerical analysis, wave theory, mathematics of flight, statistics, and mathematical meteorology. Foreword by James R. Newman.

HISTORY AND BIOGRAPHY

4:

Laki, K. The number system based on six in the proto Finno-Ugric language. J. Washington Acad. Sci. 50 (1960), no. 4, 1–11.

5:

Dyck, Martin. ★Novalis and mathematics: A study of Friedrich von Hardenberg's fragments on mathematics and its relation to magic, music, religion, philosophy, language, and literature. University of North Carolina Studies in the Germanic Languages and Literatures, No. 27. The University of North Carolina Press, Chapel Hill, N.C., 1960. xii+109 pp. Paperbound: $3.50.

A scholarly study of the influence of mathematics and pseudo-mathematics on this eighteenth century German Romantic poet and philosopher, who made no contributions to mathematics, but in whose thinking mathematical fetishism and mysticism were central.

K. O. May (Northfield, Minn.)

6:

Anonymous. Memorial to David van Dantzig: van Dantzig as a significist. Synthese 11 (1959), 319–328.

7:

Koksma, J. F. In memoriam David van Dantzig. Synthese 11 (1959), 329–334.

8:

Hemelrijk, J. David van Dantzig's statistical work. Synthese 11 (1959), 335–351.

9:

Walker, Helen M. The contributions of Karl Pearson. J. Amer. Statist. Assoc. 53 (1958), 11–22.

This is an excellent brief account of Karl Pearson's development and career as a scientist and of his contributions to science, especially to the science of statistics in which our debt to him is well summarized. In this the characterization of Pearson as a man is also well done.

C. C. Craig (Ann Arbor, Mich.)

10:

Stouffer, Samuel A. Karl Pearson—an appreciation on the 100th anniversary of his birth. J. Amer. Statist. Assoc. 53 (1958), 23–27.

This recounts personal impressions of Pearson gained by the author when, as a young student of sociology, he spent a year at University College in 1931. He then briefly discusses the impact of Pearson and his contributions on the development of sociology as a science.

C. C. Craig (Ann Arbor, Mich.)

LOGIC AND FOUNDATIONS

See also B518.

11:

Becker, Oskar. ★Grösse und Grenze der mathematischen Denkweise. Studium Universale. Verlag Karl Alber, Freiburg-München, 1959. vi+174 pp. DM 12.80.

This little book is a study of some of the problems which have, at one time or another, cast doubt on the power of the mathematical method. Among the topics discussed are the discovery of irrational numbers, the theory of infinite sets, non-Euclidean geometry, intuitionism, and Gödel's results on completeness and consistency. The author's thesis is essentially that each such problem that has arisen has either been solved or is on the way to a solution, and that the only real limitation on the mathematical method is that it is not applicable to such fields as history, where events must be understood rather than merely ordered.

The book appears to have been written for the intelligent layman. Such a reader will find much of it very difficult. Mathematicians who have a taste for the philosophical aspects of their subject but are not specialists in the foundations of mathematics, will find much in it to interest them. *John Dyer-Bennet* (Northfield, Minn.)

12:

Engeler, Erwin. Eine Konstruktion von Modellerweiterungen. Z. Math. Logik Grundlagen Math. **5** (1959), 126–131.

In dieser Arbeit wird der Robinsonsche Satz, daß ein unendliches Modell einer Menge elementarer Axiome (d.h., die Axiome sind im elementaren Prädikatenkalkül formulierbar) Erweiterungen beliebiger Kardinalzahl besitzt, durch eine transfinite "Konstruktion" bewiesen. Der Beweis macht von den Eigenschaften des Prädikatenkalküls keinen Gebrauch, insbesondere nicht von seiner Vollständigkeit. Die Axiome werden in der Skolemschen Normalform

$$(x_1)(x_2)\cdots(Ey_1)(Ey_2)\cdots F(x_1, x_2, \cdots, y_1, y_2, \cdots)$$

ohne freie Variable angenommen. Sind $R_0, R_1, R_2, \cdots$ die Prädikatensymbole, die in den Axiomen auftreten, so sind die Modelle Relationalsysteme, bestehend aus einer Menge A (der Basis) und Funktionen $f_0, f_1, f_2, \cdots$ über A, so daß f_ι dieselbe Stellenzahl wie R_ι hat und als Werte nur 0 und 1 auftreten. In einer scharfsinnigen Beweisführung reduziert Verf. das Problem auf das folgende mengentheoretische Lemma: Sind $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_\xi, \cdots$ für $\xi<\lambda$ (λ Limeszahl) Relationalsysteme mit den Basen $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\xi \subseteq \cdots$, dann gibt es ein Relationalsystem $\mathfrak{A}_\lambda$ mit der Basis $A_\lambda = \bigcup_\xi A_\xi$, so daß es für jede endliche Teilmenge E von A_λ beliebig große Ordinalzahlen $\xi<\lambda$ gibt, für die die Restriktionen auf E von $\mathfrak{A}_\xi$ und $\mathfrak{A}_\lambda$ übereinstimmen.

Dieses Lemma wird im ersten Teil der Arbeit bewiesen mit Hilfe eines auch an sich interessanten Hilfssatzes: Ist $\mathfrak{f}$ eine Familie von Funktionen in einer Menge M, derart daß (1) mit einer Funktion jede Restriktion zu $\mathfrak{f}$ gehört, (2) für jede endliche Menge $E \subseteq M$ eine Funktion in $\mathfrak{f}$ mit E als Argumentbereich existiert, und (3) jede Funktion zu $\mathfrak{f}$ gehört, deren sämtliche Restriktionen auf endliche Mengen zu $\mathfrak{f}$ gehören, dann existiert in $\mathfrak{f}$ eine Funktion mit M als Argumentbereich.

P. Lorenzen (Kiel)

13:

Friedberg, Richard M. Three theorems on recursive enumeration. I. Decomposition. II. Maximal set. III. Enumeration without duplication. J. Symb. Logic **23** (1958), 309–316.

This paper contains the solutions to two problems of the reviewer [same J. **21** (1956), 215], namely, "Is every non-recursive recursively enumerable set the union of two disjoint non-recursive recursively enumerable sets?" and "Does there exist a simple set every recursively enumerable superset of which differs either from it or from the set of all non-negative integers by only finitely many elements?", and to a problem of S. Tennenbaum, namely, "Is the class of all recursively enumerable sets recursively enumerable without repetition?" The answer to all three problems turns out to be affirmative; the first one is practically trivial (despite the fact that many people have worked on it without success!) but the two others involve complicated constructions in the typical style of the author. It is to be conjectured that eventually a common lemma will be discovered which will systematize once for all the "priority ordering" method by which Friedberg proved both these two theorems and the existence of recursively enumerable sets of incomparable degrees; certainly one has a feeling of déjà vu in reading these proofs. These remarks are not in the least intended to belittle the author's ingenuity, for which the reviewer's admiration is unbounded; the formulation and proof of such a lemma would in all probability be highly non-trivial.

{To answer a natural question, the third result can be readily generalized to the following: Every definite recursively enumerable class of recursively enumerable sets (and every infinite recursively enumerable family of partial recursive functions) can be enumerated without repetition. This was pointed out to the reviewer by Dr. Marion Pour-El.} *J. Myhill* (Berkeley, Calif.)

14:

Lorenzen, P.; Myhill, J. Constructive definition of certain analytic sets of numbers. J. Symb. Logic **24** (1959), 37–49.

The authors provide a definition of hyperarithmetical relations and sets similar to the definitions of recursive relations and sets given by Kleene [Trans. Amer. Math. Soc. **53** (1943), 41–73; MR **4**, 126] and Post [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29]. A recursive set is there defined as a recursively enumerable set whose complement is also recursively enumerable. It is shown in the present work that recursively enumerable sets and relations may be defined in terms of predicates built up from a set of atomic formulas with the aid of conjunction, disjunction, and existential quantification over numerical variables. If universal quantification is allowed, then a larger class of relations and sets, here called 'positive-inductive', is definable. Examples of positive-inductive sets that are not recursively enumerable are provided, and it is shown that a hyperarithmetical set is a positive-inductive set whose complement is also positive-inductive. The results are applied to the system of Fitch [same J. **13** (1948), 95–106; MR **9**, 559] and to explicate the gap between sets predicatively defined in the sense of Addison and Kleene [Proc. Amer. Math. Soc. **8** (1957), 1002–1006; MR **19**, 934] and classically defined sets.

E. J. Cogan (Bronxville, N.Y.)

15:

Schach, Arthur. Two forms of mathematical induction. Math. Mag. **32** (1958), 83–85.

Remarks on the equivalence between "weak" and "strong" induction.

SET THEORY

16:

Kurepa, Georges. Sur une proposition de la théorie des ensembles. C. R. Acad. Sci. Paris **249** (1959), 2698–2699.

An ordered set E is called degenerate if for each element x in E the set of points of E comparable with x is a chain in E. Let G denote the generalized continuum hypothesis, Z the axiom of choice, and γ the statement that each infinite ramified table has the same power as one of its degenerate subsets. It is shown that G, Z, and γ being true simultaneously is equivalent to the assertion that for each infinite ramified table T, the power P_DT is the immediate successor of the power of T. (P_DT is the family of all degenerate subsets of T.)

S. Ginsburg (Santa Monica, Calif.)

17:

Robertson, A. P.; Weston, J. D. A general basis theorem. Proc. Edinburgh Math. Soc. **11** (1958/59), 139–141.

Consider an abstract set X and a class $\mathscr{R}$ of finite subsets of X satisfying the following elimination axiom: If E and F are distinct members of $\mathscr{R}$ and $x \in E \cap F$ then there exists a $G \in \mathscr{R}$ such that $G \subset E \cup F$ but G does not contain x. A nonempty subset of X is said to be pure if it contains no member of $\mathscr{R}$. The main result: any two maximal pure sets have the same cardinal number of elements. In a suitable interpretation of $\mathscr{R}$, this yields the basis theorem for linear spaces (pure = linearly independent), for Abelian groups and other similar results. If an equivalence relation is defined on X and $\mathscr{R}$ is taken as the system of all couples of equivalent elements, we obtain the pigeon-hole principle.

V. Pták (Prague)

COMBINATORIAL ANALYSIS

18:

Harary, Frank. The number of functional digraphs. Math. Ann. **138** (1959), 203–210.

A directed graph (digraph) D is called "functional" if for each point b of D there is just one directed line leading away from b; functional digraphs with n points differ from functional relations on n-element sets only in not allowing lines from b to b. This paper uses an adaptation of Pólya's fundamental treatment [Acta Math. **68** (1937), 145–254] to give in compact symbolic form a generating function for the number of inequivalent functional digraphs. The author points out that the reviewer's methods [Proc. Amer. Math. Soc. **4** (1953), 486–495; MR **14**, 1053] giving a more explicit formula for the number of functional relations also apply to digraphs, but with his approach the author can derive additional generating functions for functional digraphs with varying kinds of connectedness, and also for the number of nonisomorphic functional relations.

R. L. Davis (Chapel Hill, N.C.)

ORDER, LATTICES

See also 16, 221.

19:

Benado, Mihail. Bemerkungen zur Theorie der Vielverbände. Math. Nachr. **20** (1959), 1–16.

Continuing his study of multilattices [Czechoslovak Math. J. **5 (80)** (1955), 308–344; MR **17**, 937], the author first proves that a certain set of conditions (essentially, the existence of a valuation function) suffices for a multilattice which is a {doubly-}directed set to be a metric space. Discussed as an example is the set of all angles $<180°$ on a plane. In the second chapter, it is shown that a distributive {doubly-}directed multilattice with both chain conditions is finite. The third chapter shows that every distributive complemented multilattice of finite length is a finite Boolean algebra. Some open questions are listed. {See also two papers appearing after the 1956 reception date of the present paper: Benado, C. R. Acad. Sci. Paris **246** (1958), 863–865 [MR **20** #6370]; Jakubík, Czechoslovak Math. J. **6 (81)** (1956), 426–430 [MR **20** #6369].}

P. M. Whitman (Silver Spring, Md.)

THEORY OF NUMBERS

20:

Carlitz, L. Arithmetic properties of generalized Bernoulli numbers. J. Reine Angew. Math. **202** (1959), 174–182.

Let f be a fixed integer $\geqq 1$ and $\chi(r)$ a primitive character mod f. Leopoldt [Abh. Math. Sem. Univ. Hamburg **22** (1958), 131–140; MR **19**, 1161] has defined numbers B_χ^n, $B_\chi^n(x)$ by means of

$$\sum_{r=1}^{f} \chi(r)\,\frac{te^{(r+x)t}}{e^{ft}-1} = \sum_{n=1}^{\infty} B_\chi^n(x)\,\frac{t^n}{n!}, \quad B_\chi^n = B_\chi^n(0).$$

For $f=1$, χ is the principal character, and B_χ^n reduces to the ordinary Bernoulli numbers; for $f=4$ and χ the non-principal character (mod 4), B_χ^n reduces to $-\frac{1}{2}(n+1)E_{n+1}$, where E_{n+1} is an Euler number. In the present paper the author first extends one of Leopoldt's results by showing that a theorem of the Staudt-Clausen type holds for the number $n^{-1}B_\chi^n$; he also establishes the related result that if p is a rational prime such that $p^e|n$ but $p\nmid f$, then p^e divides the numerator of B_χ^n. Next he derives congruences of the Kummer type for B_χ^n. Finally he proves that if $p^{e-1}(p-1)|m$ but $p\nmid f$, then

$$(*) \qquad \frac{1}{m+1}\,B_\chi^{m+1} \equiv \frac{1}{f}\,(1-\chi(p)) \sum_{r=1}^{f} r\chi(r) \pmod{p^e}.$$

In particular for $f=4$, (*) reduces to a known result for Euler numbers.

A. L. Whiteman (Princeton, N.J.)

21:

Cohen, Eckford. Representations of even functions (mod r). III. Special topics. Duke Math. J. **26** (1959), 491–500.

This paper concludes a series of three papers [same J. **25** (1958), 401–421; **26** (1959), 165–182; MR **20** #5756;

21 #2616] about even arithmetic functions (mod r). Here the author points out that even functions $f(n, r)$ can be used to define a discrete random variable, and using the Chebyshev theorem he discusses the estimation of $f(n, r)$ by the average

$$A(f(n, r)) = \sum_{a\,(\mathrm{mod}\, r)} f(a, r)/r$$

and by the mean value

$$\phi(n, r) = \sum_{a=1}^{n} f(a, r)/n.$$

Different even functions are considered for the illustration of the theory. *H. Bergström* (Göteborg)

22:

Isemanger, K. R. The complete factorization of $2^{132}+1$. Math. Comput. **14** (1960), 73–74.

23:

Golubev, V. A. Eine Verallgemeinerung des Primzahlensatzes von Dirichlet. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 1, 105–113 (1958). (Russian. Bulgarian and German summaries)

Zusammenfassung des Autors: "Die Arbeit enthält einige Resultate über die sogenannten Zwillinge, d. h. solche Primzahlen p, für welche auch $p+2$ eine Primzahl ist. Die Zwillinge in der arithmetischen Progression $210m+a$ sind von $x_1=11$ bis $x_2=8.10^6$ tabuliert. Verf. erweitert Euler's Tabelle der Primzahlen von der Form a^2+1 und der Primzahlen, die Teiler von Zahlen solcher Form sind, und berichtigt einige Fehler anderer Autoren."

24:

Cassels, J. W. S. Arithmetic on curves of genus 1. I. On a conjecture of Selmer. J. Reine Angew. Math. **202** (1959), 52–99.

The subject of this paper is not quite so wide as the title would indicate. For all but one brief section the discussion is restricted to points (x, y, z) on $x^3+y^3+dz^3=0$ where the ground-field K is not of characteristic 3 and contains a non-trivial cube root of unity. In section 2 a discussion is given of the algebraic background of the subject and of the link with Weil's proof of the Mordell-Weiss finite basis theorem, but the rest of the paper is independent of this section. The author believes that the methods used for the special cubic are capable of generalization along the line he indicates.

The paper investigates the circumstances which will permit the conclusion that the curve has a point over K from the fact that the curve has a point over every p-adic completion of K. The basic method is the consideration of homomorphisms of a group structure defined on points of the curve by the choice of $(1, -1, 0)$ as neutral point together with the principle of linear equivalence to various subgroups of $K^*/(K^*)^3$ defined by the curve. In developing this theory and the reciprocity theorems which follow from it the author is able to generalize the methods of descent used by E. S. Selmer in a number of papers in Acta Math. and Math. Scand. beginning with his memoir "The Diophantine equation $ax^3+by^3+cz^3=0$" [Acta Math. **85** (1951), 203–362; MR **13**, 13]. In an appendix the author takes as an example $d=5610$; he is able to show the absence of non-trivial points by discussing only the field $R(\rho)$ where R is the rational field and $\rho^2+\rho+1=0$, while Selmer had to consider 40 fields in this case. A conjecture of Selmer regarding the number of generators of the groups involved is also shown to be true [Math. Scand. **2** (1954), 49–54; MR **16**, 14].

The following theorem (VII) is chosen as reasonably typical while not requiring extended definitions. Let m_1, m_2 be in K^* but not in $(K^*)^3$. Suppose that there are points (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) defined over K on $m_j^{-1}X_j^3+m_jY_j^3+dZ_j^3=0$ $(j=1, 2)$. Then m_2 is the norm of an element of $K(m_1^{1/3})$.

J. D. Swift (Los Angeles, Calif.)

25:

Skolem, Th.; Chowla, S.; Lewis, D. J. The diophantine equation $2^{n+2}-7=x^2$ and related problems. Proc. Amer. Math. Soc. **10** (1959), 663–669.

Ramanujan [*Collected papers*, Cambridge Univ. Press, 1927, p. 327] conjectured that the above equality has no rational integral solutions for n and x except for $n=1, 2, 3, 5, 13$. Using Skolem's p-adic method [8te Skand. Mat. Kongr. Forh., Stockholm, 1934, pp. 163–188] the authors prove this conjecture and some related results. Consider the sequence $\{a_n\}$ for which $a_0=a_1=1$; $a_n=a_{n-1}-2a_{n-2}$ for $n\geqq 2$. Then $a_{n-1}^2=1$ exactly for those values of n such that $2^{n+2}-7=x^2$ has a solution. An integer appears in the sequence $\{a_n\}$ at most three times.

J. F. Koksma (Amsterdam)

26:

Birch, B. J. Note on a problem of Erdös. Proc. Cambridge Philos. Soc. **55** (1959), 370–373.

The following result is proved: given any positive, coprime integers p, q, there exists a number $N(p, q)$ such that every $n>N(p, q)$ is expressible as a sum of the form

$$n = p^{a_1}q^{b_1}+p^{a_2}q^{b_2}+\cdots,$$

where the (a_i, b_i) are distinct pairs of positive integers. The proof is self-contained and elementary, but far from trivial. *L. Mirsky* (Sheffield)

27:

Barrucand, Pierre. Sur certaines fonctions à caractère arithmétique. C. R. Acad. Sci. Paris **249** (1959), 2146–2148.

Let $r_s(n)$ be the number of representations of n as a sum of s squares. By adroit manipulation of the formula

$$\sum_{n=0}^{\infty} r_s(n)\frac{t^n}{n!} = \frac{1}{2\pi i}\int e^{tx}\Big(\sum_{k=-\infty}^{\infty} x^{-k^2}\Big)^s\frac{dx}{x},$$

in which the contour of integration is a circle $|x|=a>1$ in the complex plane, the author derives a number of interesting relations, among them

$$r_s(m) = \pi\sum_{n=0}^{\infty} r_s(n)J_{s/2-1}(\pi m, \pi n, \pi/2)$$

in which

$$J_\nu(t, k, \varphi) = \frac{1}{2\pi i}\int_{e^{-i\varphi}}^{e^{i\varphi}}\exp(tx-k/x)x^{-\nu-1}dx,$$

and the path of integration passes to the right of the origin. The last integral represents a kind of incomplete Bessel function. *A. Erdélyi* (Pasadena, Calif.)

28:

Davenport, H.; Ridout, D. Indefinite quadratic forms. Proc. London Math. Soc. (3) **9** (1959), 544–555.

Let $Q(x_1, \cdots, x_n)$ be an indefinite quadratic form in n variables with real coefficients. Suppose that when Q is expressed as a sum of squares of real linear forms with positive and negative signs, there are r positive signs and $n-r$ negative signs. The authors use results and extend methods of Birch, Davenport, Ridout and Oppenheimer to prove the following theorem: For any real indefinite quadratic form in 21 or more variables, the inequality

$$|Q(x_1, \cdots, x_n)| < \varepsilon$$

is solvable for arbitrary positive ε, in integers $x_1, \cdots, x_n$ not all zero. Further, if the coefficients of the form are not all in rational ratios, then the values assumed by the form for integers $x_1, \cdots, x_n$ are everywhere dense.

B. W. Jones (Boulder, Colo.)

29:

Mitrović, Dragiša. Le théorème Ω relatif aux dérivées de la fonction $1/\zeta$ de Riemann. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 115–119. (Serbo-Croatian summary)

It was proved by H. Bohr [see Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon, Oxford, 1951; MR **13**, 741; Theorem 8.7] that for every $\sigma > 1$ we have

$$\sup_{-\infty<t<\infty} |1/\zeta(\sigma+it)| = \zeta(\sigma)/\zeta(2\sigma).$$

This means that for $N \geqq 1$, $\varepsilon > 0$, we can choose t such that $|\mu(n)n^{-it} - |\mu(n)|| < \varepsilon$ for $n = 1, \cdots, N$. It follows that for the kth derivative ($k = 1, 2, \cdots$) we have

$$\sup_{-\infty<t<\infty} |(1/\zeta(\sigma+it))^{(k)}| = |(\zeta(\sigma)/\zeta(2\sigma))^{(k)}|.$$

N. G. de Bruijn (Amsterdam)

30:

Birman, Abraham. On the existence of a co-relation between two unsolved problems in number theory. Riveon Lematematika **13** (1959), 17–19.

The author relates the problems of the infinitude of primes of the form $4x^2+1$ and the infinitude of twin primes. This is done by showing that (1) the number $4x^2+1$ is prime if and only if x^2 cannot be expressed as the sum of a perfect square and the product of two positive consecutive integers, i.e., $x^2 \neq y^2 + z(z+1)$; (2) except for certain trivial cases, the numbers $p = 2x-1$ and $p+2 = 2x+1$ are both primes if and only if x^2 cannot be expressed as the difference of a square and the product of two consecutive integers, i.e., $x^2 \neq y^2 - z(z+1)$.

W. H. Simons (Vancouver, B.C.)

31:

Härtter, Erich. Basen für Gitterpunktmengen. J. Reine Angew. Math. **202** (1959), 153–170.

The author generalizes various theorems on bases of sets of integers to bases of sets of lattice points. The problems considered are: Lower and upper bounds and asymptotic inequalities for suitably defined counting functions for bases of order h; the number of elements in a minimal basis of order h for the set of all lattice points in a rectangular parallelepiped; and minimal bases for infinite sets of lattice points. The author also gives a sufficient condition for a set not to be a basis. The proofs consist in a fairly straightforward reduction of theorems on sets of lattice points to analogous theorems (due to Stöhr) on sets of integers.

H. B. Mann (Columbus, Ohio)

32:

Volkmann, Bodo. Bemerkung über Gitterbasen. J. Reine Angew. Math. **202** (1959), 171–173.

Generalizing a construction of Raĭkov [Mat. Sb. (N.S.) **2** (44) (1937), 595–597] and Stöhr [Math. Z. **42** (1937), 739–743] the author constructs a basis B of order h of the k-dimensional lattice vectors with non-negative coordinates, which satisfies the inequality

$$\limsup\,(B(x)/x^{k/h}) \leqq h(2^{k/h+k/(2^k-1)})(2^k-1)/ke \log 2,$$

where $B(x)$ is the number of vectors in B whose coordinates do not exceed x.

H. B. Mann (Columbus, Ohio)

33:

Rajeswara Rao, K. V. On integers a_n relatively prime to $f(a_n)$. J. London Math. Soc. **34** (1959), 145–152.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive integers functionally related by $f(a_n) = b_n$. Fairly complicated conditions on a_n and f which imply that the 'probability' of $(a_n, b_n) = 1$ is $6/\pi^2$ are derived. These results generalize those of Lambek and Moser [Canad. J. Math. **7** (1955), 155–158; MR **16**, 904]. Counterexamples are given to show that the author's conditions on a_n cannot be much weakened. Weaker conditions on f which insure the required result have recently been given by Erdös and Lorentz [Acta Arith. **5** (1958), 35–44; MR **21** #37].

L. Moser (Edmonton, Alta.)

34:

Bodmer, W. F. The limiting frequencies of integers with a given partitional characteristic. J. Roy. Statist. Soc. Ser. B **21** (1959), 134–143.

Let n be a positive integer and let $q(n)$ be the quotient of n by its largest square-free divisor. The decomposition of $q(n)$ into powers of primes may be arranged so that like powers are grouped together. This gives a characteristic partition Q of the total number of prime factors of $q(n)$. Thus for n a number like $2^4 5^2 7^4 29$ we have $q(n) = (2\cdot 7)^3 \cdot 5$ and the partition $(3^2 1)$ of the seven prime factors of $q(n)$. The paper considers the problem of calculating the density $N(Q)$ of the set of integers n that lead to a given partition Q. Thus $N(3^2 1) = .000027$. Again, for the null partition, $q(n) = 1$, $N(0) = .607927 = 6\pi^{-2}$, which is the familiar density of square-free integers. By a general method involving symmetric functions the author has computed all densities $N(Q)$ which exceed 10^{-5}. As a test the 330 numbers from 8446 to 8775 were classified and tallied and the results compared with the expected ones. The sum d_k of $N(Q)$ over all partitions Q of k is also given for $k = 0(1)5$. This function was first studied by Rényi [Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 163–165; MR **17**, 944].

D. H. Lehmer (Berkeley, Calif.)

35:

Polosuev, A. M. Uniform distribution of a system of functions representing a solution of linear first order difference equations. Dokl. Akad. Nauk SSSR **123** (1958), 405–406. (Russian)

A difference equation $\Phi(x+1)=A\Phi(x)$, where $\Phi(x)$ is an n-component vector $(\phi_1(x), \cdots, \phi_n(x))$ and A is an n by n constant matrix, is considered. Two theorems are given specifying conditions under which the points $\Phi(x)$ determined by a solution of the system in the Φ-space for $x=1, 2, 3, \cdots$ are uniformly distributed, modulo 1. It is stated that both theorems can be proved by the method used by A. G. Postnikov in Vestnik Leningrad. Univ. **12** (1957), no. 13, 81–88 [MR **21** #666].

H. L. Turrittin (Minneapolis, Minn.)

36:

Sudan, Gabriel. Sur un théorème de Obrechkoff. C. R. Acad. Sci. Paris **249** (1959), 2700–2701.

Obrechkoff used Farey series in order to prove the following theorem: Given $\omega>0$ and two integers a and n such that $0<\omega\leqq a<n$, there exists at least one pair of non-negative integers x, y, satisfying simultaneously $|\omega x-y|\leqq\left(\left[\frac{n-a}{a+1}\right]+2\right)^{-1}$ and $0<x+y\leqq n$. The present paper proves the statement using elementary geometric considerations.

E. Grosswald (Princeton, N.J.)

37:

Watson, G. L. The inhomogeneous minimum of an indefinite quadratic form. Proc. Cambridge Philos. Soc. **55** (1959), 368–370.

The author deals with the indefinite n-ary quadratic form $f(x_1, x_2, \cdots, x_n)$ and defines $M_1(f)$ as

$$\sup_{x_1^*,\cdots,x_n^*}\{\inf_{x_i\equiv x_i^* \pmod 1}|f(x_1, \cdots, x_n)|\},$$

where x_i^* range independently over all real numbers. He proves the theorem: Suppose that the quadratic form $f=f(x_1, \cdots, x_n)$ has integral coefficients and is indefinite, non-degenerate and primitive. Let ε be any positive constant, and write

$$\theta_n=\frac{3n-4}{4(n-1)^2}.$$

Then we have

$$M_1(f)\ll|d(f)|^{\theta_n+\varepsilon},$$

where $d(f)$ is the discriminant of f and the constant implied by the notation (that of Vinogradov) depends only on n and ε.

His proof is modeled on one of Birch and is related to results of Davenport, Ridout, Birch and others.

B. W. Jones (Boulder, Colo.)

FIELDS

38:

Lehti, Raimo. Evaluation matrices for polynomials in Galois fields. Soc. Sci. Fenn. Comment. Phys.-Math. **22** (1959), no. 3, 18 pp.

Let G be a Galois field $GF(p^n)$; let T be a mapping from $G\times G\times\cdots\times G$ (k factors) into G. Then T can be realized by a polynomial in n variables. This polynomial is constructed by a satisfactorily explicit process, and examples are given.

J. L. Brenner (Palo Alto, Calif.)

39:

Higman, D. G.; McLaughlin, J. E. Finiteness of class numbers of representations of algebras over function fields. Michigan Math. J. **6** (1959), 401–404.

Let $\mathfrak{o}$ be a Dedekind ring with quotient field k and let A be an algebra with unit of finite dimension over k. An order $\mathfrak{O}$ is a finitely-generated $\mathfrak{o}$-submodule of A which is a ring generating A over k and which contains the unit element of A. Left and right $\mathfrak{O}$-ideals are defined as usual. Two left ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be equivalent if $\mathfrak{a}=\mathfrak{b}\lambda$ for some $\lambda\in A$. Similarly for right ideals. It is known that the cardinal numbers of the right ideal classes and left ideal classes are the same if A is a Frobenius algebra. {The authors seem to imply that semisimplicity alone suffices. So far as the reviewer is aware, this has never been established without some extra assumption such as separability.} The proof of finiteness of the class number in the case where k is an algebraic number field has been given by Zassenhaus. The authors give a proof for the case where k is a function field of one variable with finite constant field. {The reviewer would like to propose investigation of the equality, or otherwise, of the right and left class numbers in the non-separable case.}

W. E. Jenner (Lewisburg, Pa.)

40:

Lamprecht, Erich. Restabbildungen von Divisoren. II. Arch. Math. **10** (1959), 428–437.

In this paper the author continues his studies on the reduction of a field of algebraic functions A of one variable x over the coefficient field K with respect to a prolongation p of a discrete valuation p_0 of K. Emphasis is placed on regular valuations p, i.e., the residue class field $\bar{K}$ of K with respect to p_0 is the precise coefficient field of the residue class field $\bar{A}$ of A modulo p, p is unramified over p_0, and A and $\bar{A}$ have the same genus. As the author has shown previously [part I, same Arch. **8** (1957), 255–264; MR **20** #847] the mapping A to $\bar{A}$ gives rise to a mapping of the divisor group $\{\mathfrak{a}\}$ of A into the divisor group $\{\bar{\mathfrak{a}}\}$ of $\bar{A}$. In this note the author shows how the decomposition of the image $\bar{\mathfrak{a}}$ as a product of prime divisors of $\bar{A}$—a prime divisor $\mathfrak{p}$ need not have a prime divisor for its image—can be studied (i) by representing $\mathfrak{a}$ by an ideal $\mathfrak{A}$ in the integral closure R_x of $K[x]$ in A, (ii) the structure of $R_x/\mathfrak{A}$, and (iii) the effects of residuation of the latter algebra modulo p_0.

The author points out by means of examples how the study of (iii) above resembles in many phases the study of the decomposition of a prime divisor of A/K upon extension of K; he also carefully notes aspects of (iii) which have no analogues in the theory of coefficient extensions.

O. F. G. Schilling (Chicago, Ill.)

41:

Greenspan, Bernard. A bound for the orders of the components of a system of algebraic difference equations. Pacific J. Math. **9** (1959), 473–486.

Let Φ be a system of non-zero difference polynomials in the indeterminates $y_1, \cdots, y_n$ with coefficients in an inversive difference field of characteristic zero. Let $\{\Phi\}$ be the perfect difference ideal generated by Φ. The author considers the well-known representation $\{\Phi\}=\Lambda_1\cap\Lambda_2\cap\cdots\cap\Lambda_s$, where each Λ_j is a prime ideal and none of the Λ_j contains any other. Assuming that every Λ_j is of

dimension zero (i.e., contains for each k a non-zero difference polynomial in y_k alone) the author shows that an upper bound on the order (i.e., algebraic dimension) of each Λ_i is $r_1+r_2+\cdots+r_n$, where r_k is the maximum order of the transforms of y_k appearing in Φ. This theorem is analogous to a result of J. F. Ritt [*Differential algebra*, Amer. Math. Soc. Colloq. Publ., Vol. 33, Amer. Math. Soc., New York, 1950; MR **12**, 7] for differential polynomials. The author sharpens his (the author's) theorem in the special case where Φ has exactly n elements, and sharpens it still further in the case where Φ has exactly n elements and $n=2$.
W. Strodt (New York)

ALGEBRAIC GEOMETRY

See also 24, 238.

42:

Samuel, Pierre. Elements of algebraic geometry. Notas Mat. No. 18 (1959), 114 pp. (Portuguese)

These notes had their origin in a series of lectures delivered by the author in 1958 at the Instituto de Matematica Pura y Aplicada, Río de Janeiro, and written up by A. Azevedo and A. Micali. Chapter I treats specializations, places and valuations. Chapter II deals with irreducible and noetherian spaces (descending chain condition for closed sets). Chapters III and IV present affine and projective varieties and cover such topics as regular extensions, dimension and components of intersection of algebraic sets. Chapter V studies rational, birational, regular and biregular transformations, the quotient of a variety with respect to a finite group of automorphisms, symmetric products, Segre varieties and monoidal transformations.

Many of the concepts are well illustrated by examples. There are some minor misprints but they can easily be corrected by the reader. This book should be very useful to readers with some knowledge of the basic topics in algebra.
E. Lluis (Mexico City)

43:

Serre, J.-P. On the fundamental group of a unirational variety. J. London Math. Soc. **34** (1959), 481–484.

Since any known numerical invariants distinguish unirational varieties from rational varieties, a question is raised as to whether the fundamental group does so. A negative answer is given here, by showing that a projective non-singular unirational variety V is simply connected. A tentative extension is made to the case of characteristic p, by defining a projective non-singular variety V^n to be unirational if there is a surjective rational separable map f from the projective space P^n to V. Showing the existence of the maximum unramified covering V' of V, which turns out to be unirational, the author proves that V has no Severi torsion if one could prove $\chi(V')=1$.
T. Matsusaka (Evanston, Ill.)

44:

Snapper, Ernst. Multiples of divisors. J. Math. Mech. **8** (1959), 967–992.

Let X be a projective variety. Let $D_1, \cdots, D_n$ be divisors of X which are locally linearly equivalent to zero everywhere on X. For the divisor $D=\sum m_iD_i$ (m_i being rational integers), $L(D)$ denotes the sheaf associated with D (i.e., the sheaf whose stalks are local ideals for D); when F is an algebraic sheaf on X, $F(D)$ denotes the sheaf $L(D)\otimes F$. $\chi(X, F(D))$ denotes the Euler-Poincaré characteristic of $F(D)$ (i.e., $\chi(X, F(D))=\sum_i(-1)^ih^i(X, F(D))$). The main result in this paper is the following: The Euler-Poincaré characteristic $\chi(X, F(m_1D_1+\cdots+m_nD_n))$ is a polynomial in $m_1, \cdots, m_n$ with rational coefficients, and the degree of the polynomial does not exceed the dimension of the support of F.

The proof is based on the following lemma (Proposition 9.1). Assume that a function q from the n-ple product Z^n of the set Z of rational integers into the set Q of rational numbers satisfies the following property: there is an integer d such that for every i ($1\leq i\leq n$) and for any fixed $m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_n \in Z$ the function $q(m_1, \cdots, m_{i-1}, z_i, m_{i+1}, \cdots, m_n)$ is a polynomial in z_i of degree at most d. Then q is a polynomial in $z_1, \cdots, z_n$ with rational coefficients.
M. Nagata (Kyoto)

LINEAR ALGEBRA

See also 176, B400.

45:

Newman, Morris. Kantorovich's inequality. J. Res. Nat. Bur. Standards Sect. B **64B** (1960), 33–34.

Let A be a hermitian matrix of order n all of whose eigenvalues lie on the closed interval $\alpha\leq t\leq\beta$; let $x_1, \cdots, x_k$ be vectors such that $\sum_{i=1}^k \|x_i\|^2=1$, the norm being euclidean; let $f_1, f_2, \cdots, f_m$ be positive convex functions on the above interval, and let $c_1, \cdots, c_m$ be positive numbers satisfying $c_1c_2\cdots c_m=1$. Then

$$\prod_{i=1}^m \Big[\sum_{j=1}^m (f_i(A)x_j, x_j)\Big]^{1/m} \leq \frac{1}{m}\max\Big[\sum_{i=1}^m c_if_i(\alpha), \sum_{i=1}^m c_if_i(\beta)\Big].$$

The paper gives the proof for the case $k=1$, $m=2$, and concludes by stating the more general theorem due to Ky Fan. The Kantorovich inequality [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 6 (28), 89–185; MR **10**, 380] arises for $\alpha>0$, $f_1(t)=t$, $f_2(t)=1/t$, $c_1=(\alpha\beta)^{-1/2}$.

{In the final equation on p. 33, the right hand side should be a sum instead of a product.}
A. S. Householder (Oak Ridge, Tenn.)

46:

Parodi, Maurice. Sur les matrices stochastiques. C. R. Acad. Sci. Paris **249** (1959), 1436–1437.

The reviewer [Duke Math. J. **19** (1952), 75–91; MR **13**, 813] proved a number of theorems on the characteristic roots of stochastic matrices. The author gives a simple proof for some of these results.
A. Brauer (Chapel Hill, N.C.)

47:

Marcus, Marvin; Khan, N. A. Space of k-commutative matrices. J. Res. Nat. Bur. Standards Sect. B **64B** (1960), 51–54.

Let A and X be real or complex $N\times N$-matrices. Then the commutator operation $[A, X]=AX-XA=T(X)$ defines a linear homogeneous transformation in the N^2-dimensional matrix space M_N. One is interested in the

subspace $\mathfrak{A}_k$ of all matrices $X \in M_N$ that satisfy the condition $T^k(X) = [A, X]_k = [A, [A, X]_{k-1}] = 0$. These X are said to be k-commuting with A. It is found that the dimension of $\mathfrak{A}_k$ can be expressed in terms of the degrees of the elementary divisors of A if each root of A occurs in not more than one elementary divisor. For the investigation A is assumed in its Jordan canonical form, i.e., as direct sum $J_1 + \cdots + J_q$ of the Jordan blocks corresponding to distinct roots of A, so that also $X = X_1 + \cdots + X_q$ with the X_s of the same size as the J_s. The explicit result is too complicated for quotation. It is also stated that if A has more than one elementary divisor for the same root, then a simple formula for dim $\mathfrak{A}_k$ in terms of the degrees of the elementary divisors does not seem to exist. There is a lemma which makes the computation of dim $\mathfrak{A}_k$ in each case possible. A second theorem states bounds for dim $\mathfrak{A}_k$ if $k \geq 2(N-q)+1$.

H. Schwerdtfeger (Montreal)

48:

Bellman, Richard. Kronecker products and the second method of Lyapunov. Math. Nachr. **20** (1959), 17–19.

The second method of Lyapunov for the solution of problems concerning stability involves the solution of the matrix equation $AX + XA' = C$ for the unknown symmetric matrix X. In this note the author uses the concepts of Kronecker products and sums of matrices to give a new proof of the known fact that a unique solution X exists for every C if and only if $\lambda_i + \lambda_j \neq 0$, $(i, j = 1, \cdots, n)$, where the λ's are the characteristic roots of A. The proof shows also that the equation $AX + XB = C$ has a unique symmetric solution X for every C if and only if $\lambda_i + \mu_j \neq 0$ $(i, j = 1, \cdots, n)$, where the λ's are as above, and the μ's are the characteristic roots of B.

L. A. MacColl (New York)

49:

Duffin, R. J. An analysis of the Wang algebra of networks. Trans. Amer. Math. Soc. **93** (1959), 114–131.

An algebra for which $x + x = x^2 = 0$ for each x is called a Wang algebra—it is a Grassman algebra over the field of integers modulo 2. The author surveys some of the pertinent results on Grassman algebras and justifies the techniques in network theory based on the Wang algebra. The most interesting results center around the notion of "just" subspaces and just discriminants. Let U be a real vector space with a given base $e_1, \cdots, e_n$. A vector is just if its coordinates in this base are 0 or ± 1's. Let V be a subspace of U, $p_i = \sum_{j=1}^n k_{ij} e_j$ $(i = 1, \cdots, m)$ a base for V. Let K be the matrix with elements k_{ij} and G a diagonal matrix with elements $g_1, \cdots, g_n$. The function

$$f(g_1, \cdots, g_n) = KGK_t$$

is called the discriminant of G in V. Finally let the exterior (outer) product of the p_i be $p_1 \cdots p_m = \sum d_t e_{t_1} e_{t_2} \cdots e_{t_m}$. V is a just subspace if the d_t have values 0 or ± 1. Then V is just if and only if the coefficients of the discriminant are 0's and 1's and if and only if the just vectors in V are, under addition modulo 2, a vector space of the same dimension as V. For example, in a network, the cycles—or currents satisfying the Kirchoff laws—form a just subspace. *J. B. Giever* (University Park, N.M.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 39, 62, 63.

50:

Szász, F. Sur les anneaux infinis ne contenant que des sous-anneaux nontriviaux de l'index fini. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 1, 29–33 (1958). (Bulgarian and Russian summaries)

Those infinite rings which as additive groups are cyclic are characterized as those infinite rings each non-trivial subring of which has finite index. See the author, Publ. Math. Debrecen 4 (1956), 237–238 [MR **18**, 187] for another characterization. *F. Haimo* (St. Louis, Mo.)

51:

Isbell, J. R. On the multiplicative semigroup of a commutative ring. Proc. Amer. Math. Soc. **10** (1959), 908–909.

This paper is devoted to a proof of the theorem: If the multiplicative semigroup of a commutative ring is finitely generated, it is finite. *C. E. Rickart* (New Haven, Conn.)

52:

Szép, J. Über eine neue Erweiterung von Ringen. II. Acta Sci. Math. Szeged **20** (1959), 202–214.

[For part I, see same Acta **19** (1958), 51–62; MR **20** #4578.] If R is a ring and A and B are two subrings with $A + B = R$, $A \cap B = \emptyset$, we write $R = A + B$. Given A and B as rings to start with, the author in an earlier paper has solved the extension problem of finding all R with $R = A + B$. In this paper he considers his extension theory in detail for three cases: A and B both nil-rings, A and B both skew fields, and $a \cdot b = b \cdot a$ for all $a \in A$, $b \in B$. He proves for instance that if $R = A + B$ and both A and B are skew fields then at least two of the following are true: A is a left ideal, A is a right ideal, B is a left ideal, B is a right ideal. Other similar results are established.

D. K. Harrison (Philadelphia, Pa.)

53:

Thierrin, G. Sur le radical corpoïdal d'un anneau. Canad. J. Math. **12** (1960), 101–106.

An ideal C of a ring A is called field-like (corpoïdal) if A/C is a (not necessarily commutative) field. The intersection of all the field-like ideals of A is called the field-like radical of A. Several characterizations of the latter are given. If T is the union of all the subsets H of A such that $abH = baH$ for all $a, b \in A$, then T is an ideal of A and the field-like radical of T is the intersection of T with the Perlis-Jacobson radical of A. Thus, if A is semisimple, and $T \neq 0$, then T is a subdirect sum of fields.

M. Henriksen (Lafayette, Ind.)

54:

Lesieur, L.; Croisot, R. Sur les anneaux premiers noethériens à gauche. Ann. Sci. École Norm. Sup. (3) **76** (1959), 161–183.

The authors study a prime ring A under one-sided chain conditions introduced by A. W. Goldie [Proc. London Math. Soc. (3) 8 (1958), 589–608; MR **21** #1988], namely ($1l$) every direct sum of nonzero left ideals of A contains only a finite number of terms, and ($2l$) the ascending chain

condition holds for left annihilator ideals. These are assumed throughout. Both hold in a left Noetherian ring. A left ideal X is closed if for every $b \in X$, there exists $x \in (b|$ such that $x \neq 0$ and $X \cap (x| = 0$. The closed left ideals form a modular complemented lattice of finite length and they coincide with the complemented left ideals of Goldie. The minimal closed left ideals coincide with Goldie's basic ideals. A prime ring is an integral domain if and only if the ideal (0) is $\cap$-irreducible. The main result is the following one-sided generalisation of Goldie's theorem: A prime ring A has a left quotient ring $Q(A)$, which is a complete matrix ring over a division ring, if and only if (1l) and (2l) hold. The left quotient ring $Q(A)$ is the injective envelope of A as an A-module.

D. C. Murdoch (Vancouver, B.C.)

55:

Piccinini, Renzo. Observations on local rings. Soc. Parana. Mat. Anuário (2) **1** (1958), 32–34. (Portuguese. English summary)

Exposé de quelques propriétés connues de la topologie ordinaire des anneaux locaux. *E. Lluis* (Mexico City)

56:

Lafon, Jean P. Quelques résultats sur le dual d'un module de type fini sur un anneau commutatif et applications à l'étude des modules tels que leurs anneaux d'endomorphismes soient commutatifs. C. R. Acad. Sci. Paris **249** (1959), 1849–1851.

Let E be a finitely generated module over a commutative Noetherian ring A with unit element such that the ring $L(E)$ of all A-endomorphisms of E equals A. Then E is isomorphic to an ideal of A if and only if there exists a homomorphism h of E into A such that $h(E)$ is faithful. Also, the canonical map $E \rightarrow E \otimes_A K$ is a monomorphism where K is the total ring of quotients of A. The case when A is a local ring is also considered.

L. Fuchs (Budapest)

57:

Nagahara, Takasi; Nobusawa, Nobuo; Tominaga, Hisao. Galois theory of simple rings. IV. Math. J. Okayama Univ. **8** (1958), 189–194.

[For part III, see Nobusawa and Tominaga, same J. **7** (1957), 163–167; MR **20** #3185.] The extension of automorphisms is proved for simple rings with finiteness assumptions that are weaker than in Tominaga, same J. **6** (1957), 153–170 [MR **19**, 382]. Specifically, let R be a simple ring with minimum condition, let S be a simple subring, and let S_1 be the ring obtained by adjoining to S a full set of matrix units of R. If (1 and 2) R is Galois over both S and S_1 and locally finite over S, if (3) the centralizer V of S in R is finite over its own center, if (4) R has finite or countable dimension over the double centralizer of S, and if R_1 and R_2 are regular subrings of R containing S and with V finite over the R-centralizer of each R_i, then every isomorphism over S of R_1 onto R_2 is extendable to an automorphism of R, and R is Galois and locally Galois over R_i. It is also shown that the set of conditions (1), (2), (3) is preserved when S is replaced by a regular subring of R which is finite over S. *D. Zelinsky* (Evanston, Ill.)

58:

Bourne, Samuel. On the radical of a positive semiring. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1519.

Two definitions of the radical in certain semirings [Słowikowski and Zawadowski, Fund. Math. **42** (1955), 215–231; MR **18**, 223; Bourne, same Proc. **37** (1951), 163–170; MR **13**, 7; cf. p. 166] are equivalent.

E. Hewitt (Seattle, Wash.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

59:

Cohn, P. M. Two embedding theorems for Jordan algebras. Proc. London Math. Soc. (3) **9** (1959), 503–524.

If A is an associative algebra over a field of characteristic $\neq 2$, with multiplication $a \cdot b$, the algebra defined on the vector space of A by $ab = \frac{1}{2}(a \cdot b + b \cdot a)$ is a Jordan algebra that will be denoted by $J_1(A)$; if, moreover, A has an involution $*$, the Jordan algebra of self-adjoint elements of A will be denoted by $J_2(A, *)$ and $J_2(A, *) \subseteq J_1(A)$. A special Jordan algebra of the form $J_1(A)$ or $J_2(A, *)$ will be called a J_i-algebra ($i = 1, 2$).

The author defines abstract J_1-algebras as follows: A J_1-algebra is a commutative linear algebra K with a binary operation $[a, b]$ defined on K which is bilinear and antisymmetric and satisfies the identities $[a, b^2] = 2[a, b]b$, $[a, [b, c]] = (ab)c - (ac)b$. He then shows: (a) Every J_1-algebra is of the form $J_1(A)$, for a suitable associative algebra A; (b) a Jordan algebra is special if and only if it can be embedded in a J_1-algebra. Likewise a variety of J_2-algebras may be defined abstractly as a commutative linear algebra K with an additional quadrilinear antisymmetric operation $[a, b, c, d]$ satisfying certain identities. Again, it is shown that every J_2-algebra is of the form $J_2(A, *)$ for a suitable associative algebra A and that a Jordan algebra J is special if and only if it can be embedded in an algebra of the form $J_2(A, *)$. A considerable portion of the paper is devoted to showing that there exist a finite number of identities defining the variety of J_2-algebras. *L. J. Paige* (Los Angeles, Calif.)

60:

Patterson, E. M. Note on non-associative rings with regular automorphisms. J. London Math. Soc. **34** (1959), 457–464.

The author considers regular automorphisms (those which leave no non-zero element fixed) of non-associative rings of prime or zero characteristic. Since any such ring can be embedded in a non-associative algebra A over an algebraically closed field F in such a way that a regular automorphism of the ring can be extended to a regular automorphism of the algebra, the results are stated for such algebras. If T is a regular automorphism whose minimal equation is

$$(T-\lambda_1)^{r_1}(T-\lambda_2)^{r_2}\cdots(T-\lambda_m)^{r_m} = 0,$$

where $\lambda_1, \cdots, \lambda_m$ are distinct elements of F and if the minimal equation of T on the derived algebra A_2 is

$$(T-\lambda_1)^{s_1}(T-\lambda_2)^{s_2}\cdots(T-\lambda_m)^{s_m} = 0,$$

where $0 \leq s_i \leq r_i$; then, if $s_i \neq 0$, there exist λ_j, λ_k ($1 \leq j \leq m$, $1 \leq k \leq m$) such that $\lambda_i = \lambda_j \lambda_k$ and $r_j + r_k - 1 \geq s_i$. It is shown that algebras satisfying the conditions of the above result do exist. Let L be the set whose elements are $\lambda_1, \cdots, \lambda_m$ and $\alpha(L)$ the class of algebras each of which admits a regular automorphism T whose minimal equation is as in the above theorem. The terms 'product', 'nilpotent of class p', 'soluble of index p', and 'simple' are defined for L and it is proved that the algebras in $\alpha(L)$ are nilpotent of class at most p [soluble of index at most p] if and only if L is nilpotent of class p [soluble of index p]. Also if L is non-simple, every algebra in $\alpha(L)$ is non-simple. There is also a classification of regular automorphisms of degree 2, 3, or 4 of non-associative rings of prime or zero characteristic satisfying $R^2 = R$. *L. A. Kokoris* (Chicago, Ill.)

HOMOLOGICAL ALGEBRA

61:

Beršteĭn, I. Geometric dimension of abelian groups. Rev. Math. Pures Appl. **3** (1958), 93–99. (Russian)

Let the group Π be the direct limit of a sequence of groups Π_n, and let K_n be a free acyclic Π_n-complex. The author constructs from these a free acyclic Π-complex K such that $\dim K \leq \sup \dim K_n + 1$. Since by a theorem of Eilenberg and Ganea [Ann. of Math. (2) **65** (1957), 517–518; MR **19**, 52] the minimal dimension for a free acyclic Π-complex may be identified with the cohomological dimension, $\dim \Pi$, of the group Π if $\dim \Pi \geq 3$, the above construction yields that $\dim \Pi \leq \sup \dim \Pi_n + 1$ if $\sup \dim \Pi_n \geq 3$, and a similar estimate for the category holds. In case Π is an abelian torsion-free non-finitely generated group of finite rank the sharper result $\dim \Pi = \operatorname{rank} \Pi + 1$ is obtained. *W. T. van Est* (Leiden)

62:

Higman, D. G. On isomorphisms of orders. Michigan Math. J. **6** (1959), 255–257.

Let $\mathfrak{o}$ be a (commutative) integral domain with quotient field k, and $\mathfrak{O}$ an $\mathfrak{o}$-algebra. We denote by $I(\mathfrak{O})$—the depth of $\mathfrak{O}$—the ideal of $\mathfrak{o}$ consisting of the elements of $\mathfrak{o}$ which annihilate the cohomology groups $H^1(\mathfrak{O}, T)$ for all two-sided $\mathfrak{O}$-modules T. $\mathfrak{O}$ is called an $\mathfrak{o}$-order if it is finitely generated and torsion-free as an $\mathfrak{o}$-module; if moreover $\mathfrak{O} \otimes_{\mathfrak{o}} k$ is a separable k-algebra then we call $\mathfrak{O}$ separable. A necessary and sufficient condition for an $\mathfrak{o}$-order $\mathfrak{O}$ to be separable is that $I(\mathfrak{O}) \neq 0$ [the author, Canad. J. Math. **7** (1955), 509–515; MR **19**, 527]. The author proves the theorem that if $\mathfrak{o}$ is a complete discrete valuation ring with maximal ideal $\mathfrak{p}$, a separable $\mathfrak{o}$-order $\mathfrak{O}$ of depth $\mathfrak{p}^s$ is isomorphic with an $\mathfrak{o}$-order $\mathfrak{Q}$ if and only if $\mathfrak{o}$- (or $\mathfrak{o}/\mathfrak{p}^{2s+1}$-) algebras $\mathfrak{O}/\mathfrak{p}^{2s+1}\mathfrak{O}$ and $\mathfrak{Q}/\mathfrak{p}^{2s+1}\mathfrak{Q}$ are isomorphic. From this it is immediately derived, for example, that if a complete discrete valuation ring $\mathfrak{o}$ has a finite residue class field there are only finitely many non-isomorphic separable $\mathfrak{o}$-orders of given finite rank and depth.

G. Azumaya (Sapporo)

63:

Higman, D. G. On representations of orders over Dedekind domains. Canad. J. Math. **12** (1960), 107–125.

This paper generalizes results on integral representations of groups (due to Brauer, Maranda, and Reiner) to the case of integral representations of an order. In this review it is possible to give only a sample of the types of theorems which the author obtains.

Let $\mathfrak{o}$ be an integral domain with quotient field k; all $\mathfrak{o}$-modules are assumed to be finitely-generated and torsion-free, and all short exact sequences of $\mathfrak{o}$-modules are assumed to split over $\mathfrak{o}$. The 'rational hull' of an $\mathfrak{o}$-module M is defined as $M \otimes_{\mathfrak{o}} k$. Let $\mathfrak{O}$ be an $\mathfrak{o}$-algebra with rational hull A; a 'principal' (right) $\mathfrak{O}$-module is one whose rational hull is a direct sum of right ideal components of A. It is proved that an $\mathfrak{O}$-module M is principal if and only if $D(M) \neq 0$, where $D(M)$ consists of all $x \in \mathfrak{o}$ such that $x \cdot \operatorname{Ext}^1(M, N) = 0$ for all $\mathfrak{O}$-modules N. The author gives a new proof of his earlier criterion that A be separable.

Turning to the case where $\mathfrak{o}$ is a local ring with maximal ideal $\mathfrak{p}$, let M be an $\mathfrak{O}$-module and set $D(M) = \mathfrak{p}^s$, where we take $s = \infty$ when $D(M) = 0$. Call the exponent s the 'depth' of M. Generalizing a theorem of Maranda, the author shows that two principal modules of depth s are isomorphic if and only if they are isomorphic modulo $\mathfrak{p}^{s+1}$. If $\mathfrak{o}/\mathfrak{p}^{s+1}$ is finite, the number of non-isomorphic $\mathfrak{O}$-modules of depth s and fixed $\mathfrak{o}$-rank is finite. Depth is preserved under passage to the completion of $\mathfrak{o}$.

Now let $\mathfrak{o}$ be a complete local ring. For M an $\mathfrak{o}$-module set $M^{(s)} = M/\mathfrak{p}^s M$, and in particular let $\bar{M} = M/\mathfrak{p}M$. Extending a result due to Reiner, the author shows that $\pi^s \operatorname{Ext}^1(M^{(s+1)}, N^{(s+1)}) = 0$ implies $\pi^s \operatorname{Ext}^1(M, N) = 0$. He deduces that M is $\mathfrak{O}$-projective if and only if $\bar{M}$ is $\bar{\mathfrak{O}}$-projective. Further, $\mathfrak{O}$ has depth zero if and only if $\bar{\mathfrak{O}}$ is a separable $\bar{\mathfrak{o}}$-algebra. The author gives a new proof of a theorem of Brauer's: Every projective $\mathfrak{O}$-module is a direct sum of indecomposable summands which are uniquely determined up to isomorphism and order of occurrence.

Finally let $\mathfrak{o}$ be a Dedekind ring. The author obtains relations between the global and local theories. For example, let M be an $\mathfrak{O}$-module, and let $d_{\mathfrak{p}}(M)$ be its depth at $\mathfrak{p}$; then

$$D(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{d_{\mathfrak{p}}(M)}.$$

This is applied to the problem of determining the number of genera in the set of $\mathfrak{O}$-modules with given rational hull, and this number is proved finite under certain natural hypotheses.

This paper is unfortunately not self-contained since it relies heavily on earlier definitions and notations which are not given explicitly in the present paper.

I. Reiner (Urbana, Ill.)

GROUPS AND GENERALIZATIONS

See also 79, 80, 196, B253.

64:

Charles, Bernard. Une caractérisation des intersections de sous-groupes divisibles. C. R. Acad. Sci. Paris **250** (1960), 256–257.

Problem 2 of the reviewer's book *Abelian groups* [Hungarian Acad. Sci., Budapest, 1958; MR **21** #5672] is solved. It is shown that a subgroup H of an abelian divisible group G may be represented as the intersection of divisible subgroups of G if and only if, for each prime p, either the socles of the p-components of G and H are different or every element of H is divisible by p in H. {Reviewer's note: The more general problem arises how to characterize the subgroups of an abelian group G that are intersections of pure subgroups of G.}

L. Fuchs (Budapest)

65:

Sands, A. D. The factorization of abelian groups. Quart. J. Math. Oxford Ser. (2) **10** (1959), 81–91.

If A, B are subsets of a group G and every element $g \in G$ can be uniquely represented as $g = ab$ ($a \in A$, $b \in B$), then $G = AB$ is a factorization of G. A subset H of G is periodic if there is an element $g \in G$, $g \neq e$, such that $Hg = H$. Hajós's theorem on finite abelian groups gave rise to the following problem: If $G = AB$ is a factorization of a finite abelian group G, does it follow that either A or B is periodic? G is a "good" or "bad" group according as the answer to this question is affirmative or negative for G. There exist "bad" groups (even among the cyclic groups), and on the other hand some groups are known to be "good".

In this paper the non-cyclic groups are considered and the question is settled for some of those 13 classes of non-cyclic groups for which this problem has not been decided [cf. de Bruijn, Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 258–264; MR **15**, 8]. Moreover in the last section extensions of previous results on finite abelian groups to certain infinite abelian groups are made. One of the results: If G is the direct product of groups of type $\{p_i^{\lambda_i}\}$, where $i = 1, 2, \cdots, k$, the numbers p_i are distinct primes and the exponents λ_i are positive integers or infinity, $AB = G$, and the number of elements of A is a power of a prime, then either A or B is periodic.

J. Erdős (Debrecen)

66:

Páter, Z. Des groupes qui coïncident avec leur groupe commutateur. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 245–247. (Romanian. Russian and French summaries)

Some elementary remarks on groups described by the title. *B. Harris* (Evanston, Ill.)

67:

Dlab, Vlastimil. Some relations among the generating systems of abelian groups. Czechoslovak Math. J. **9 (84)** (1959), 161–171. (Russian. English summary)

In a previous paper of the author [same J. **8 (83)** (1958), 54–61; MR **20** #1707] the notions of reducible, strongly reducible, hereditarily reducible, and hereditarily strongly reducible generating systems of an abelian group are introduced. In this way one can discern six types of generating systems (e.g., hereditarily reducible generating system which is not strongly reducible). Any abelian group determines a combination of these types (the set of types of its generating systems). In the present paper the author answers the question: which combinations of types can be determined by an abelian group? It turns out that only eight combinations satisfy this condition.

J. Erdős (Debrecen)

68:

Crouch, R. B. A class of irreducible systems of generators for infinite symmetric groups. Proc. Amer. Math. Soc. **10** (1959), 910–911.

Let S be the group of those permutations of the positive integers which displace only a finite number of integers, and A its unique subgroup of index 2. Let $n_0 = 1, n_1, n_2, \cdots$ be a strictly increasing sequence of positive integers, let c_i be the cycle $(n_i, n_i + 1, \cdots, n_{i+1})$, and let M be the set of all these cycles. The group generated by M is triply transitive and contains the 3-cycle $c_0^{-1}c_1^{-1}c_0c_1$, and so it contains A. Thus it is A if all the integers $n_{i+1} - n_i$ are even, and S if at least one of them is odd. If any c_i is omitted from M, the group generated by the remainder is not transitive, so that M is an irreducible set of generators of S or A. Thus, for each of these groups, the set of its irreducible systems of generators has the power of the continuum.

Graham Higman (Oxford)

69:

Bachman, G. On finite nilpotent groups. Canad. J. Math. **12** (1960), 68–72.

The author establishes necessary and sufficient conditions that all groups of given order n be nilpotent. If $n = \prod p_i^{\alpha_i}$, the conditions are that p_i does not divide $(p_j - 1)(p_j^2 - 1) \cdots (p_j^{\alpha_j} - 1)$ for any i, j. This generalizes the conditions, due to Dickson [Trans. Amer. Math. Soc. **6** (1905), 198–207; p. 201], that all groups of order n be Abelian ($\alpha_j \leqq 2$ and $p_i \nmid (p_j^{\alpha_j} - 1)$). *P. J. Higgins* (London)

70:

Gorenstein, Daniel. Finite groups which admit an automorphism with few orbits. Canad. J. Math. **12** (1960), 73–100.

The author studies a finite group G with an automorphism ϕ, such that, for a fixed element g and a fixed integer r, every element of G can be written as

$$\phi^i(g\phi^r(g) \cdots \phi^{r(j-1)}(g))$$

for suitable integers i, j. His main theorems are (i) that G is soluble, and (ii) that if ϕ has no fixed elements other than the identity, G is nilpotent of class at most 2.

Graham Higman (Oxford)

71:

Gorenstein, Daniel; Herstein, I. N. On the structure of certain factorizable groups. I. Proc. Amer. Math. Soc. **10** (1959), 940–945.

If the finite group G has the factorisation AB, where A, B are cyclic, and A is its own normaliser, then the derived group G' is cyclic, $G = AG'$ and $A \cap G' = 1$.

Graham Higman (Oxford)

72:

Lefebvre, Pierre. Sur certains théorèmes d'isomorphisme pour les demi-groupes. C. R. Acad. Sci. Paris **249** (1959), 1995–1997.

Analogues for semi-groups of the isomorphism theorems of group theory are numerous. This paper extends the list by considering the homomorphisms of a semi-group onto a group with zero adjoined. In particular, the author is concerned with the lattice properties of the family of those sub-semi-groups of a semi-group D each of which is the kernel of a homomorphism of D on a group with zero.

R. S. Pierce (Seattle, Wash.)

73:

Ross, Kenneth A. A note on extending semicharacters on semigroups. Proc. Amer. Math. Soc. **10** (1959), 579–583.

A bounded complex function χ on a semigroup G is called a semicharacter of G if $\chi(x)\neq 0$ for some $x\in G$ and $\chi(xy)=\chi(x)\cdot\chi(y)$ for all $x, y\in G$ [E. Hewitt and H. S. Zuckerman, Acta Math. **93** (1955), 67–119; MR **17**, 1048; Št. Schwarz, Czechoslovak Math. J. **4** (79) (1954), 219–247; MR **16**, 1085]. Write $a|b$ if there is an $x\in G$ such that $ax=b$. The following is the main result of the paper. Let S be a subsemigroup of the commutative semigroup G and χ a semicharacter on S. χ is extendable to a semicharacter on G if and only if the following condition holds: $a|b$ and $a, b\in S$ imply $|\chi(a)|\geq|\chi(b)|$. *Št. Schwarz* (Bratislava)

74:

Preston, G. B. Embedding any semigroup in a $\mathscr{D}$-simple semigroup. Trans. Amer. Math. Soc. **93** (1959), 351–355.

In the semigroup S, let L_a and R_a respectively be the smallest left and right ideals containing a. Let $\mathscr{D}$ be the equivalence generated by the set of pairs (a, b) for which either $L_a=L_b$ or $R_a=R_b$. Then S is called $\mathscr{D}$-simple if $\mathscr{D}=S^2$. Such a semigroup is necessarily simple. It is shown that every semigroup can be embedded in a $\mathscr{D}$-simple semigroup. The proof uses an elegant characterization of the $\mathscr{D}$-equivalence classes in the semigroup of mappings of a set into itself. *R. S. Pierce* (Seattle, Wash.)

75:

Murata, Kentaro. On P-components of normal ideals in a semigroup. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 1–7.

Let S be a semigroup with identity element. The notions of an order o of S, of left [right] o-ideals of S, v-ideals and a number of other notions are introduced. The definitions parallel those given, for instance, in chapter 6 of Jacobson's *Theory of rings* [Amer. Math. Soc., New York, 1943; MR **5**, 31]. The groupoid of v-ideals defined on a system of equivalent (in a well-defined sense) maximal orders is studied, the main purpose being to transfer some results concerning the groupoid of normal ideals in a ring (proved by K. Asano) to the case of semigroups.

Št. Schwarz (Bratislava)

76:

Yamada, Miyuki. Note of idempotent semigroups. V. Implications of two variables. Proc. Japan Acad. **34** (1958), 668–671.

[For part IV, see Kimura, same Proc. **34** (1958), 121–123; MR **20** #4604.] Elements of the free semigroup generated by two "variables" x, y will be denoted by $f(x, y)$, $g(x, y)$, $\cdots$. An equality is a relation of the form $f(x, y)=g(x, y)$. A relation defined by $\{f_i=g_i|i\in I\}\Rightarrow\{f_j^*=g_j^*|j\in J\}$ is called an implication. Let S be an idempotent semigroup. If an equality [implication] holds whenever we regard x, y as any elements $\in S$ we shall say that the equality [implication] is satisfied by S. Main results: Every family of equalities on S is equivalent (in a well-defined sense) to one of 16 distinct classes which are explicitly described. Every implication on S is equivalent (again in a well-defined sense) to one of 8 explicitly given implications. Proofs are omitted. *Št. Schwarz* (Bratislava)

77:

Sade, Albert. Solution générale de l'équation de transitivité sur un ensemble quelconque avec division bilatère. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 451–455.

Suppose that four quasigroups Q_1, Q_2, Q_3, Q_4 defined on the same set E with operations ϕ_1, ϕ_2, ϕ_3, ϕ_4 are such that $(x\phi_1y)\phi_2(z\phi_3y)=x\phi_4z$ for all x, y, $z\in E$. Then it is shown that Q_1, Q_2, Q_3', Q_4 are all isotopic (in specified ways) to the same group, where Q_3' is the quasigroup "parastrophic" to Q_3, obtained by $x\phi_3y=z\Leftrightarrow x=y\otimes z$. {Compare Stein's observation [Trans. Amer. Math. Soc. **85** (1957), 228–256; MR **20** #922; Corollary 5.6] that a quasigroup obeying the transitive law $xy\cdot zy=xz$ is parastrophic—or, in Stein's nomenclature, conjugate—to a group.}

I. M. H. Etherington (Edinburgh)

78:

Cowell, W. R. Loops with adjoints. Canad. J. Math. **12** (1960), 134–144.

Denote by $N(G)$ the usual 3-net associated with a loop $(G, +)$. A permutation σ on G is called an adjoint of G if $\sigma-\varepsilon$, where ε is the identity mapping, is a permutation on G. The author shows that a set of points of $N(G)$ can be adjoined as a line to $N(G)$ (in the usual incidence sense) if and only if G possesses an adjoint σ. The incidence system consisting of $N(G)$ together with the adjoined lines associated with a set Σ of adjoints of G is called the quasinet $Q(G, \Sigma)$. In order to lend respectability to $Q(G, \Sigma)$ as an object of mathematical interest, the author imposes restrictions on the set Σ. For example, a permutation σ is called linear if $\sigma=\delta\rho(a)$, where δ is an automorphism of G and $\rho(a): x\to x+a$; then an adjoint is linear if both σ and $\sigma-\varepsilon$ are linear as permutations. If a linear adjoint σ has $a=0$, σ is an automorphism adjoint.

The paper continues with a geometric interpretation to quasinets of well-known results of loop theory. The following results are typical. (i) If the inverse property loop G contains a characteristic normal subloop H of index 2, then G has no automorphism adjoints. (ii) Let G be a loop with a set Σ of adjoints so that the quasinet $Q(G, \Sigma)$ is an affine plane. Then Σ is irreducible; that is, G has no proper Σ-subloops. *L. J. Paige* (Los Angeles, Calif.)

TOPOLOGICAL GROUPS AND LIE THEORY

79:

Steinberg, Robert. Variations on a theme of Chevalley. Pacific J. Math. **9** (1959), 875–891.

The paper contains the discovery of two families of new simple groups, one of which seems to coincide with the family given independently by J. Tits [Séminaire Bourbaki, 10e année: 1957/1958, Secrétariat mathématique, Paris, 1958; MR **21** #4981] from a different point of view. Roughly speaking, the groups are constructed by a method which generalizes that of constructing certain unitary groups.

Let $\mathfrak{g}$ be a simple Lie algebra over the complex field, and K a field. For each root r of $\mathfrak{g}$ and $t\in K$ define the automorphism $x_r(t)$ of the algebra $\mathfrak{g}_K$ as in Chevalley, Tôhoku Math. J. (2) **7** (1955), 14–66 [MR **17**, 457]. Let $\mathfrak{U}$ [resp. $\mathfrak{B}$] be the group generated by the $x_r(t)$ with $r>0$ [resp. $r<0$], and G the group generated by $\mathfrak{U}$ and $\mathfrak{B}$. If $\mathfrak{g}$ is of type

(A_l), (D_l), or (E_6), the Dynkin diagram of $\mathfrak{g}$ shows that the root system admits an automorphism $r\to\bar{r}$ of order 2 such that $\bar{r}>0$ whenever $r>0$; G admits a corresponding automorphism σ which maps $x_r(t)$ upon $x_{\bar{r}}(\pm\bar{t})$, where $t\to\bar{t}$ is an automorphism of order 2 of K. Let $\mathfrak{U}^1$ [resp. $\mathfrak{B}^1$] be the group of elements in $\mathfrak{U}$ [resp. $\mathfrak{B}$] invariant by σ, and G^1 the group generated by $\mathfrak{U}^1$ and $\mathfrak{B}^1$. Then G^1 for (E_6) turn out to be new simple groups. If $\mathfrak{g}$ is of type (D_4), then the root system admits an automorphism of order 3 which, combined with an automorphism of K of order 3, again yields new simple groups. Another family of (new) infinite simple groups is also obtained from (D_4). The orders of the new finite simple groups are computed in an ingenious manner. *Rimhak Ree* (Vancouver, B.C.)

80:

Kostant, Bertram. A formula for the multiplicity of a weight. Trans. Amer. Math. Soc. **93** (1959), 53–73.

Weyl's well-known formula for the characters of semisimple groups contains as coefficients the multiplicities of the different weights. For an effective computation of the multiplicities Weyl's formula cannot be used. The reviewer [Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 369–376; MR **16**, 673] derived the formula

$$(1)\qquad m_\lambda(\nu)((\lambda+\delta,\lambda+\delta)-(\nu+\delta,\nu+\delta)) = \sum m_\lambda(\nu+p\alpha)(\nu+p\alpha,\alpha),$$

where $m_\lambda(\nu)$ is the multiplicity of the weight ν in the representation with highest weight λ, δ is the half sum of the positive root forms, and the sum runs over all positive root forms α and all positive integers p. This formula is of practical use, as has been shown by the reviewer's calculations of characters of E_8 [Nederl. Akad. Wetensch. Proc. Ser. A **57** (1954), 487–491; MR **16**, 673]. Weyl's formula can be derived from it.

With a criticism of the formula (1) (see below), the author again starts from Weyl's formula, rewritten in the form

$$(2)\qquad m_\lambda(\nu) = G(\nu)-\sum \det S\cdot m_\lambda(\nu+\delta-S\delta),$$

where the sum runs over all elements $S\neq 1$ of the Weyl group and $G(\nu)=\det S_0$ if $\nu+\delta=S_0(\lambda+\delta)$, or zero, if such an S_0 does not exist. This is a recursive formula for the multiplicities. Applying this formula by iteration one gets the explicit formula

$$(3)\qquad m_\lambda(\nu) = \sum \det S\cdot Q(S(\lambda+\delta)-(\nu+\delta))$$

where $Q(\mu)$ is the "signed" number of ordered partitions of μ as a sum of expressions $\delta-S\delta$, $\mu=\sum(\delta-S_{\nu_i}\delta)$, counted in the positive or negative sense according to the sign of $\prod\det S_{\nu_i}$. {There is a misprint in the first formula on p. 69.}

Up to this point the results are only formal. The main discovery of this paper is the equivalence between the partition function Q and a much more natural partition function P that appears by a direct approach to the multiplicity problem (not using Weyl's formula). $P(\mu)$ is simply the number of (non-ordered) partitions of μ as a sum of positive root forms. The identity of P and Q is a far from trivial result, the background of which is not clear. It is proved by a highly interesting artifice. It is immediately seen that for large integers p, $m_{p\delta}(p\delta-\mu)=Q(\mu)$, because then only the contribution $S=1$ remains. The same formula with P instead of Q is proved by representation theory; the effect of the large p is to withdraw the weight from the influence of the walls of the Weyl chamber.

Putting $\lambda=0$ in (3) a recursive formula for $P=Q$ can be stated: for $\mu\neq 0$,

$$P(\mu) = -\sum_{S\neq 1}\det S\cdot P(\mu-(\delta-S\delta)).$$

{The author's criticism of the use of formula (1) can hardly be justified. Twice in his paper (after formulas 1.1.3 and 6.1.5) he asserts that that formula has the disadvantage of necessitating division by a term that might vanish. Clearly he has overlooked the fact that only dominant ν should be taken into consideration and that the dominant ν for which $m_\lambda(\nu)\neq 0$ are characterised by the property of $\lambda-\nu$ of being a sum of positive roots. So divisions by zero terms can never occur. The non-vanishing of the divisor was proved in the reviewer's paper.

On the other hand the author's formula (2) that is asserted to be superior to (1) shares the great disadvantage of Weyl's formula from which it is derived, that the terms to be computed have a huge number of summands with alternating signs, the overwhelming majority of which will finally cancel, whereas formula (1) has a small number of positive summands only. The division to be carried out when using (1) is not a disadvantage, but a real advantage because it provides an effective check (integers must result from the division). Finally (2) has the disadvantage of using complicated terms of positive root forms (the $S\delta-\delta$) instead of the root forms themselves. This means that for an effective use of (2) an extensive memory is needed. It would seem that there does not exist any electronic computer that could carry out the computations of E_8 mentioned above using formula (2), whereas with the use of formula (1) it was done by pencil and non-electrical table computer (which could have been dispensed with). To be sure, in the case of a relatively high highest weight ($\lambda\gg\delta$) formula (2) would be superior to (1). But with existing computational means, this could be effective only when the group dimension is extremely small.

These are only practical remarks. They cannot belittle the theoretical result of the equivalence of P and Q, which is really marvelous.} *H. Freudenthal* (Utrecht)

81:

Racah, G. Theory of Lie groups. Nuovo Cimento (10) **14** (1959), supplemento, 67–74.

This is an expository paper on elementary Lie group theory concentrating mostly on the basic notations. There is a last section with some remarks on the representation theory of Lie groups. *L. Auslander* (Bloomington, Ind.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

82:

Hunter, Robert P. On the semigroup structure of continua. Trans. Amer. Math. Soc. **93** (1959), 356–368.

This paper contains an investigation of the structure of topological semigroups defined on continua. The first part

of the paper deals with a semigroup S irreducible between two points with $S^2=S$. The author shows that if S has a zero then S is an arc. The latter part is devoted to semigroups on hereditarily unicoherent continua. As a corollary the author obtains the result that a 1-dimensional clan with zero is arcwise connected.

The proofs given, however, are somewhat hard to follow due to typographical errors and careless statements by the author. It is sometimes questionable if the author actually means what his statements imply. For example, in Definition 1.3, the author obviously means, though he does not state it, for the sets A, B, and C to be pairwise disjoint; for without this restriction Theorem 1.5 is false. Also the proof of Theorem 1.11 is inadequate; for the claim made that, if V is an open set containing A, the complement of a maximal prime ideal, then $S\backslash V$ must have finitely many components, is clearly false.

Anne Lester (New Orleans, La.)

83:

Hofmann, Karl Heinrich. Über archimedisch angeordnete, einseitig distributive Doppelloops. Arch. Math. **10** (1959), 348–355.

An algebraic system $(K, +, \cdot)$ with two binary operations is a doubleloop if (a) $(K, +)$ is a loop with zero element 0, (b) $(K-\{0\}, \cdot)$ is a loop with identity element 1, and (c) $x0=0x=0$ for all $x \in K$. The definitions of a right distributive doubleloop and an ordered doubleloop should be apparent.

The author shows that every ordered loop, relative to the interval topology induced by $<$, is a topological doubleloop [Math. Z. **70** (1958), 213–230; MR **21** #1364]. Moreover the set H of all x such that $x(ab)=(xa)b$ for all $a, b \in K$ is a topological subdoubleloop whose non-zero elements form a topological group. The set H enters in various structure theorems obtained for topological doubleloops and it is shown that a one-sided distributive, archimedian ordered, doubleloop is a commutative neofield which is dense in a real neofield.

L. J. Paige (Los Angeles, Calif.)

FUNCTIONS OF REAL VARIABLES

84:

McShane, Edward James; Botts, Truman Arthur. ★Real analysis. The University Series in Undergraduate Mathematics. D. Van Nostrand Company, Inc., Princeton, N.J.-Toronto-London-New York, 1959. ix+272 pp. $6.60.

The title of this book is intended to be a more accurate description of its content than the traditional and perhaps no longer appropriate "theory of a real variable". The central theme, reflecting the interest of the senior author, is integration; the preliminary chapters (0 through IV) present the essential tools for a discussion of the integral from the viewpoint of Daniell. The last chapter (VII) is a brief and condensed presentation of some of the basic ideas of functional analysis. An interesting feature of chapter II is the authors' decision to present the theory of limits in the general framework of "directions" ("nets", "directed systems") rather than to treat the sequential and "continuous" cases separately.

The text is intended for a student who has completed an undergraduate analysis sequence. In level and treatment, it is similar to the comparable book by L. M. Graves [*The theory of functions of real variables*, 2nd ed., McGraw-Hill, New York, 1956; MR **17**, 717]. Most of the standard topics are treated, and with skill and clarity. (However, the reviewer looked in vain for the concept of category.) Chapter I is devoted to a presentation of the characteristic properties of the real field. Chapters II and III deal with the basic properties of continuous functions. Chapter IV treats the notion of bounded variation and connects this with differentiation properties of functions, ending with a proof of the Implicit Function Theorem in a local form. Chapter V presents the Lebesgue-Stieltjes integral as the extension of an elementary integral defined for step functions and then extended to semi-continuous functions. Portions of measure theory are recaptured from this in the next chapter. Finally, in Chapter VII, the authors introduce the notion of a normal linear space and, with emphasis upon the L_p spaces, define linear functional and operator, prove the Hahn-Banach extension theorem, and discuss the essential role of bounded Hermitian operators in Hilbert space.

There is no commonly accepted route by which a novice meets the basic concepts of analysis; differences of opinion and of approach are therefore to be expected. The reviewer regrets the authors' infrequent use of algebraic concepts and terminology, even in topics where they are natural and unforced, and cast light upon the structure at hand; he also feels that there are numerous places where a greater felicity of notation would have rendered the treatment more lucid.

R. C. Buck (Princeton, N.J.)

85:

Menger, Karl. Rates of change and derivatives. Fund. Math. **46** (1958), 89–102.

In this paper the author uses the term "fluent" with domain A to denote a real-valued function defined on a set A, in place of the expression "variable quantity" which he used previously. The word "function" is reserved for fluents whose domain is a class of real numbers. He gives a quite general definition of the concept of the rate of change of one fluent with respect to another. The fluents may have different domains, provided a binary relation between their domains is specified, and the rate of change is then relative to this relation. It is a function of two variables rather than just one. By way of topology in the domains of the fluents, it is assumed that these are limit classes with limits determined by a continuous semi-metric. The terms limit class and semi-metric are not defined explicitly.

It is shown that the ordinary notion of the derivative of a function reduces essentially to a special case of this concept of rate of change, and that if two fluents are functionally related by means of a differentiable function, then their relative rate of change under certain circumstances is expressible in terms of the derivative of this function. The rate of change of a fluent f with respect to a fluent g is found to be on occasions equal to the reciprocal of the rate of change of g with respect to f. A theorem corresponding to the chain rule for derivatives is proved. This involves the notions of a quasi-continuous relation and of a Π-binormal pair. It is shown by examples that the relative rate of change may exist even in cases where the fluents are nowhere functionally related.

O. Frink (University Park, Pa.)

86a:

Dobrescu, Eugen; Sălăgean, Ioan. Une extension du critérium de compacité d'Arzelà. Com. Acad. R. P. Romîne **9** (1959), 215–221. (Romanian. Russian and French summaries)

86b:

Dobrescu, Eug.; Sălăgean, I. Sur les fonctions hyperboliques continues. Com. Acad. R. P. Romîne **9** (1959), 419–424. (Romanian. Russian and French summaries)

The authors study hyperbolically continuous functions defined on a two-dimensional closed interval I. Both papers study the relationships between compactness and equi-hyperbolic-continuity for sets of such functions. In addition, the first paper contains a sufficient condition for a hyperbolically continuous function to be continuous, and the second paper shows that a real-valued function f on I is hyperbolically continuous if and only if for each point $(a, b) \in I$ the function f has the form

$$f(x, y) = g(x, y) + f(x, b) + f(a, y),$$

for some continuous function g on I.

Victor Klee (*Copenhagen*)

87:

Pilika, P. Boundary functions of classes $H^{(r_1, \cdots, r_n)}_{p_1, \cdots, p_n}$. Dokl. Akad. Nauk SSSR **128** (1959), 677–679. (Russian)

The classes of the title are the generalized Lipschitz classes of Nikol'skiĭ [Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 321–336; MR **20** #6028]. The upper indices refer to orders of differentiation and the lower indices to L-classes. A boundary function is, roughly speaking, one that belongs to a given class but not to any class with larger upper indices and equal lower indices. The author constructs explicit examples of boundary functions to show that an imbedding theorem given by Nikol'skiĭ cannot be improved. *R. P. Boas, Jr.* (Evanston, Ill.)

88:

Montel, Paul. Sur la dépendance linéaire des fonctions. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 2, 99–104 (1959). (Bulgarian and Russian summaries)

In this note the author proves a necessary and sufficient condition for n continuous functions of a real variable to be dependent. The condition is expressed in terms of the vanishing of n determinants involving differences of the functions. *L. M. Graves* (Chicago, Ill.)

89:

Ciesielski, Z. Some properties of convex functions of higher orders. Ann. Polon. Math. **7** (1959), 1–7.

This note, which won first place in a student mathematical competition, describes two definitions of an n-convex, real-valued function on an open interval and proves relationships between these and boundedness, measurability, and continuity similar to results for ordinary convex functions. *M. M. Day* (Urbana, Ill.)

90:

Dobrev, D. Integraldarstellung einer Klasse von Funktionen. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 1, 43–50 (1958). (Bulgarian. Russian and German summaries)

Every function f that is convex over 2-space and homogeneous of degree 1 admits the representation

$$f(x, y) = \int_{-\infty}^{\infty} |y - sx| d\alpha(s) + A|x|,$$

where $\alpha(s)$ is nondecreasing and $\int |s| d\alpha(s) < \infty$.

R. P. Boas, Jr. (Evanston, Ill.)

91:

Rudin, Walter. Positive definite sequences and absolutely monotonic functions. Duke Math. J. **26** (1959), 617–622.

The author proves the interesting theorem (not novel in design) that if $F(x)$ is defined real valued in $(-1, 1)$, and if for any almost periodic sequence $\{a_n\}$, $n = 0, \pm 1, \pm 2, \cdots$, $-1 < q_n < 1$, the transformed sequence $\{F(a_n)\}$ is again almost periodic, then

$$F(x) = \sum_0^\infty c_n x^n, \; c_n \geq 0, \; -1 < x < 1;$$

moreover a continuous $F(x)$ is of this kind if for any constants $a \geq 0$, $b \geq 0$, $a + b < 1$,

$$\int_{-\pi}^{\pi} F(a + b \cos \theta) \cos n\theta d\theta \geq 0 \quad (n = 0, 1, 2, \cdots).$$

S. Bochner (Princeton, N.J.)

MEASURE AND INTEGRATION

92:

Bertolini, Fernando. La teoria algebrica della misura e della integrazione, e suo rapporto con la teoria classica. II. Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 259–274.

In a previous paper of the same title [same Ann. (3) **11** (1957), 225–247; MR **20** #2418], the author has shown equivalence between (a) a Caratheodory system of soma as a basis for a theory of integration and (b) a system of subsets $\mathfrak{B}$ of a space U which is a ring of sets and satisfies the condition that for any sequence of subsets B_k of $\mathfrak{B}$, there exists a maximum set (denoted $\prod_k B_k$) in $\mathfrak{B}$, contained in $\bigcap_k B_k$. In case $\prod_k B_k = \bigcap_k B_k$, the system is said to be of type (c). A system of type (b) can be extended to one of type (c) as follows: Let $\mathfrak{N}$ be the minimal class of subsets of U which satisfy the conditions: (1) if $B_k \in \mathfrak{B}$, and $\prod_k B_k = \emptyset$ then $\bigcap_k B_k \in \mathfrak{N}$; (2) if $N' \subset N \in \mathfrak{N}$, then $N' \in \mathfrak{N}$; (3) if $N_k \in \mathfrak{N}$, and $\bigcup_k N_k \subset B \in \mathfrak{B}$, then $\bigcup_k N_k \in \mathfrak{N}$. The class $\mathfrak{C}$ is then the δ-ring generated by $\mathfrak{B} \cup \mathfrak{N}$. In $\mathfrak{C}$ we get equivalence classes relative to $\mathfrak{N}$, there being a unique B of $\mathfrak{B}$ in every equivalence class in case $(U; \mathfrak{B})$ satisfies the Stone property that the product of sets in any ultrafilter is not empty. Measure functions are extended to $\mathfrak{C}$ by the condition that $\mu(N) = 0$ for $N \in \mathfrak{N}$, leading to extensions of measurable functions and of integration. Compare O. Haupt and C. Y. Pauc [Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1957**, 273–290; MR **20** #2417].

T. H. Hildebrandt (Ann Arbor, Mich.)

FUNCTIONS OF COMPLEX VARIABLES

See also 190, 191, 209, 231.

93:

Moppert, C. F. On the notion of analyticity. Proc. Amer. Math. Soc. **10** (1959), 574–576.

Let E_n denote a normed vector space over a field Γ with a bilinear operator $\otimes$. E_n is made into an algebra by specializing the operator to be a product which is distributive but not necessarily commutative or associative.

The author defines $f(x)$ to be analytic with respect to $\otimes$ at $x=x_0$ if

$$f(x_0+h) = f(x_0)+l(x_0) \otimes h+r(h)$$

such that $l(x_0)$ and $r(h)$ are in E_n and $|r(h)|=O(|h|)$. He also defines a generalization of the Cauchy-Riemann differential equations by using the multiplication constants of the algebra. *J. A. Ward* (Alamogordo, N.M.)

94:

Riabouchinsky, Dimitri P. ★Sur quelques nouvelles généralisations de la théorie des nombres complexes et leurs applications. I. Les nombres complexes généralisés et leur application en géométrie, algèbre, analyse et mécanique des fluides. Préface de Pierre Vernotte. Publ. Sci. Tech. Ministère de l'Air, no. 343, Paris, 1958. x+166 pp. 2750 francs.

The author makes a detailed study of number systems of the form $x+ey$, where x and y are real and $e^2=-1, +1$, or 0. Applications to fluid mechanics are pointed out. Extension is made to the case where e is a continuous function of x and y. The author also considers the operation of obtaining the absolute value and its inverse as well as the passage to the limit $L(x)=0$ and its inverse $X=L^{-1}(0)$. *J. A. Ward* (Alamogordo, N.M.)

95:

Pommerenke, Ch. On the derivative of a polynomial. Michigan Math. J. **6** (1959), 373–375.

Let $f_n(z)=z^n+\cdots$ be a polynomial of degree n. Assume that the region $|f_n(z)|\leq 1$ is connected. The author proves (among other results) that in this region $|f_n'(z)|\leq \frac{1}{2}en^2$. The reviewer conjectured that $|f_n'(z)|<\frac{1}{2}n^2$ is also true. *P. Erdös* (Adelaide)

96:

Flett, T. M. On the summability of a power series on its circle of convergence. Quart. J. Math. Oxford Ser. (2) **10** (1959), 179–201.

This paper has partly the nature of a survey. The author clarifies the relationships between inequalities associated with a power series belonging to the class H^p, and supplies new proofs of several known results. In particular, he gives a simple proof of the reviewer's result concerning the summability factor of the power series of H^p $(0<p<1)$ [Tôhoku Math. J. (2) **7** (1955), 96–109; ibid. **8** (1956), 125–146; MR **17**, 361; **18**, 889]. *G. Sunouchi* (Evanston, Ill.)

97:

Dvoretzky, A.; Erdös, P. Divergence of random power series. Michigan Math. J. **6** (1959), 343–347.

Let $\varphi_n(t)$ denote the Rademacher functions, and let $P(z;t)=\sum_{n\geq 0}\varphi_n(t)a_nz^n$ $(0\leq t<1)$. It is well known that if (1) $\sum|a_n|^2=\infty$, then for almost all t the series $P(z;t)$ diverges almost everywhere on $|z|=1$. The authors are interested in replacing "almost everywhere on $|z|=1$" by "everywhere on $|z|=1$" [cf. Dvoretzky, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 199–203; MR **18**, 75]. The main result of this paper is the following. Let $\{c_n\}$ be a monotone sequence of positive numbers tending to zero and satisfying

$$\text{(2)} \qquad \limsup\, (\log c_n^{-1})^{-1} \sum_{j=0}^{n} c_j^2 > 0.$$

If $|a_n|\geq c_n$ for all n, then for almost all t the series $P(z;t)$ diverges everywhere on $|z|=1$ (e.g., $c_n=n^{-1/2}$, $n>N$). Condition (2) cannot be replaced by (1). *N. J. Fine* (Princeton, N.J.)

98:

Pirl, Udo. Zum Normalformenproblem für endlich-vielfach zusammenhängende schlichte Gebiete. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **6** (1956/57), 799–802.

A classical problem in the theory of conformal mapping requires the mapping of a plane domain of finite connectivity onto a so-called normal domain whose boundary is geometrically or analytically characterized. The author makes an attempt to treat with a unified method the various mapping problems of this type. His main result indicates that the existence of practically all mappings of this kind can be proved with the help of the Koebe-Grötzsch continuity method, provided that the uniqueness of the mapping has been established. The general result is applied to two special cases previously treated by Walsh [Trans. Amer. Math. Soc. **82** (1956), 128–146; MR **18**, 290]. *O. Lehto* (Helsinki)

99:

Huber, Heinz. Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. Math. Ann. **138** (1959), 1–26.

Let $\mathfrak{F}$ be a 2-dimensional, orientable, closed, analytic Riemann surface of constant negative curvature (hyperbolic space form). Let $\mathfrak{W}$ be a homotopy class of closed paths, and $\mathfrak{P}$ a primitive homotopy class (that is, $\mathfrak{P}$ cannot be represented as a power of another homotopy class). Defining $\mu(\mathfrak{W})=\inf\int ds$ over all paths in $\mathfrak{W}$, the author investigates the asymptotic behavior of $\mu(\mathfrak{W})$ and $\mu(\mathfrak{P})$. His tools are the representation of $\mathfrak{F}$ as the quotient space of the unit disk by a Fuchsian group Λ and the spectral theory of the Laplace-Beltrami operator on $\mathfrak{F}$.

Among other results the author proves that the number of classes $\mathfrak{P}$ for which $\mu(\mathfrak{P})\leq t$ is asymptotic to e^t/t, and that the number of classes $\mathfrak{W}$ for which $\mu(\mathfrak{W})\leq t$ is also asymptotic to e^t/t. The asymptotic law is therefore the same for all closed space forms. *J. Lehner* (East Lansing, Mich.)

100:

Oikawa, Kôtaro. On a criterion for the weakness of an ideal boundary component. Pacific J. Math. **9** (1959), 1233–1238.

The definition of weakness of a boundary component γ of a Riemann surface F, as a generalisation of what Grötzsch called "vollkommen punktförmige Randkomponente" of a plane region, is stated, using exhaustions and a special type of harmonic functions in the neighbourhood of γ. With a different type of harmonic functions N. Savage [Duke Math. J. **24** (1957), 79–95; MR **18**, 647] gave a sufficient condition for weakness. The author shows that Savage's criterion is also necessary, but the exhaustion of F cannot always be of the special type called canonical. *K. Strebel* (Fribourg)

101:

Vincze, István. Transcendent entire functions of maximum modulus. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 451–459. (Hungarian)

For a transcendent entire function $F(z)$, $F(0)\neq 0$, write $S_n(r)=r^{-n}M(r)$, $M(r)=\max_{|z|=r} F(z)$, $M_n=\min_{0<r<\infty} S_n(r)$, the latter being attained for $r=r_n$. The uniqueness of r_n, and numerous other properties, are established, relying mainly on the convexity of $\log M(r)$ as a function of $\log r$. For example, $M_{n-1}/M_n \leq r_n \leq M_n/M_{n+1}$. Also found are bounds for $\int_0^\infty (r^n/M(r))dr$.

F. V. Atkinson (Toronto, Ont.)

102:

Rahman, Q. I. On connections between growth and distribution of zeros of integral functions. Canad. J. Math. **12** (1960), 40–44.

The author proves two theorems on entire functions of small order, generalizing known results by allowing wider distribution of zeros. (1) Let $f(z)=\prod (1-z/z_n)$, of order less than 1, $z_n=r_ne^{i\theta_n}$, and let

$$\sum r_n^{-\sigma}\{2 \sin (\theta_n+\pi)\sigma+\sin 2\pi\sigma\} = C,$$

finite, $0<\sigma<1$. Then $\lim r^{-\rho}n(r)=A$ is equivalent to

$$\int_0^r x^{-1-\sigma}\{\log |f(-x)|-\pi \cot \pi\sigma\cdot n(x)\}dx \sim$$
$$\pi A(\rho-\sigma)^{-1}(\cot \pi\rho-\cot \pi\sigma)r^{\rho-\sigma}+\pi C\sigma^{-1}/(1-\cos 2\pi\sigma).$$

(2) Let $n_+(r)$ and $n_-(r)$ count the zeros in the upper and lower half-planes of $f(z)$ of genus 0 and lower order λ, $0<\lambda<1$. If $\Re z_n=o(|z_n|)$, then $\lim r^{-\rho}n(r)=A$ is equivalent to $\lim x^{-\rho} \log |f(x)|=\frac{1}{2}\pi A \csc \frac{1}{2}\pi\rho$; the case when $n_-(r)$ is bounded was given by Clunie [J. London Math. Soc. **32** (1957), 138–144; MR **19**, 259].

R. P. Boas, Jr. (Evanston, Ill.)

103:

Ibragimov, I. I. Some inequalities for entire functions of finite degree in several variables. Dokl. Akad. Nauk SSSR **128** (1959), 1114–1117. (Russian)

Extensions to entire functions of several variables of the methods and inequalities of the author's paper [Uspehi Mat. Nauk **12** (1957), no. 3 (75), 323–328; MR **19**, 737]. *R. P. Boas, Jr.* (Evanston, Ill.)

104:

Shah, S. M. Meromorphic functions of finite order. Proc. Amer. Math. Soc. **10** (1959), 810–821.

Let $\prod (z)$ be a Weierstrass product of order ρ and let $n(r)$ be the number of its zeros in $|z|<r$, $M(r)$ its maximum modulus on $|z|=r$, and $T(r)=T(r,f)$ its characteristic. Then the author proved [J. London Math. Soc. **15** (1940), 23–31; MR **1**, 307] that

$$\text{(1)} \qquad \liminf_{r\to\infty} \frac{\log M(r)}{n(r)\phi(r)} = 0$$

provided that $\phi(r)$ is a positive non-decreasing function such that

$$\text{(2)} \qquad \int_1^\infty \frac{dt}{\phi(t)} < +\infty.$$

For non-integral order ρ, (1) can be sharpened to

$$\text{(3)} \qquad \liminf \frac{\log M(r)}{n(r)} \leqq B(\rho),$$

where $B(\rho)$ is a constant depending only on ρ.

The author now shows by examples that the growth condition (2) cannot be weakened in order that (1) should hold. He also obtains a number of results closely related to (1) and (3) of which the following is typical: If f is a meromorphic function of non-integral order ρ and g_1, g_2, are distinct meromorphic functions, satisfying $T(r, g_i(z))=o(T(r,f))$, $i=1, 2$, and if $n_i(r)$ denotes the number of roots of the equations $f=g_i$ in $|z|<r$, then for $k\geqq 1$

$$\liminf_{r\to\infty} \frac{T(kr,f)}{n_1(r)+n_2(r)} \leqq k^{1+[\rho]}B(\rho).$$

W. K. Hayman (London)

105:

Dunducenko, L. E.; Kasianiuk, S. A. Concerning two classes of analytic functions in a circular annulus. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **13** (1959), no. 4 (31), 85–101. (Romanian)

The authors consider two classes of functions that are regular in an annulus and establish representation formulas and sharp inequalities for members of the classes. By specialization they obtain many special classes that have been previously studied. They first define a special class B^0 of functions that are meromorphic in $q<z<1$ and represented by a structure formula involving theta functions. Then Z^0 is the class of functions f meromorphic in the same annulus, with derivatives having m specified zeros and n specified poles and subject to

$$\arg \{z^{-1}f'(z)/g'(z)\} < \varepsilon\pi/2 \quad (0 < \varepsilon \leqq 2),$$

where g belongs to B^0 and g' has the same zeros and poles as f'; and Z^* is a similar class where the zeros and poles belong to f and $\arg \{z^{-1}f(z)/g'(z)\} < \varepsilon\pi/2$. The elements of Z^0 and of Z^* are characterized by rather complicated structure formulas {which were not entirely clear to the reviewer because of misprints}. The authors obtain exact bounds for $|f'(z)|$, $|f(z)|$ and $\arg f'(z)$ or $\arg f(z)$ in each class. Special cases include classes studied by Zmorovič [Mat. Sb. (N.S.) **40** (82) (1956), 225–238; MR **18**, 648], star-shaped univalent functions, close-to-convex univalent functions, and many others.

R. P. Boas, Jr. (Evanston, Ill.)

106:

de La Vallée Poussin, C. Valeurs exceptionnelles des fonctions continues et uniformes. Fondements du théorème de Picard. Ann. Soc. Sci. Bruxelles. Sér. I **73** (1959), 295–301.

Remarks on the topological content of Picard's theorem.

L. Ahlfors (Cambridge, Mass.)

107:

Hiong, King-lai. Sur la quasi-normalité de quelques familles de fonctions holomorphes dans le cercle unité. Sci. Record (N.S.) **3** (1959), 335–341.

The author continues his series of papers, rediscovering certain parts of Nevanlinna theory. No proofs are given, presumably because they are well known.

W. K. Hayman (London)

108:

Sakaguchi, Kōichi. Some classes of multivalent functions. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **6**, 205–222 (1959).

Let $M(p, \alpha)$ denote the class of functions f satisfying either (1) or (2) below: (1) f is meromorphic in $|z| \leqq 1$ and $f(z) \neq 0, \infty$ for $|z|=1$ and the image curve of the unit circle by f cuts $2p$ times a pair of rays starting from the origin and forming an angle of magnitude α such that $0 \leqq \alpha \leqq \pi$; (2) f is meromorphic in $|z|<1$ and for every ρ $(\rho<1)$ sufficiently close to 1, the function $f(\rho z)$ has the properties as stated in (1). The functions in $M(p, \alpha)$ are said to be of order p on an average in two directions forming an angle α. We denote by $N(p, \alpha, q)$ the subclass of $M(p, \alpha)$ consisting of those functions f which are normalized as follows: (a) f has no pole in the unit circle, (b) f has no zero except at the origin, and (c) f has the expansion $f(z)=z^q+\sum_{n=q+1}^{\infty} c_n z^n$ where q is a non-negative integer. The author obtains coefficient estimates for functions f in $N(p, \alpha, q)$ which are of the form $|c_{q+1}| \leqq 2s$ and $|c_{q+2}| \leqq 2s^2+q$, where $s=p+2+\alpha/\pi$. He also obtains numerous distortion theorems and related estimates for the radii of q-valence, starlikeness, and convexity. For f in $N(1, \alpha, 1)$ with $\alpha=0$ or π, the partial sums of f are univalent and starlike with respect to the origin for $|z|<1/4s$ and univalently convex for $|z|<1/8s$, where $s=3-\alpha/\pi$. He also obtains coefficient estimates and distortion theorems for the k-fold symmetric functions in $N(p, \alpha, q)$.

G. Springer (Lawrence, Kans.)

109:

Singh, Vikramaditya. Extremum problems for the coefficient a_3 of the bounded univalent functions. Proc. London Math. Soc. (3) **9** (1959), 397–416.

Let S be the class of regular univalent functions f in $|z|<1$ such that $f(z)=a_1z+a_2z^2+a_3z^3+\cdots$, $a_1>0$, and satisfying $|f(z)|<1$ for $|z|<1$. Using an adaptation of Shiffer's method of interior variation, the author proves two theorems giving estimates for $|a_3|$. In theorem 1, for $e^{-1} \leqq a_1 \leqq 1$, one has the sharp estimate $|a_3| \leqq a_1(1-a_1^2)$, and the extremum function $w=f(z)$ satisfies $w-w^{-1}=a_1^{-1}(z-z^{-1})$. For $0 \leqq a_1 \leqq e^{-1}$, $|a_3| \leqq 2a_1(\sigma-a_1)+a_1(1-a_1^2)$, where σ is a root of $\sigma \log \sigma + a_1 = 0$, $e^{-1} \leqq \sigma \leqq 1$, and the rather complicated relation defining the extremum function is also given. For $a_1=e^{-1}$, there is a one-parameter family of extremum functions. In theorem 2, there is the requirement added to the functions $f \in S$ that the coefficients a_n all be real and that $a_2>0$. For $-2a_1^2 \log a_1 \leqq a_2 \leqq 2a_1(1-a_1)$, one has the sharp estimate

$$a_3 \leqq a_1(1-a_1)+\frac{a_2^2 \log a_1 e}{a_1 \log a_1},$$

while for $0 \leqq a_2 \leqq -2a_1^2 \log a_1$, one has

$$a_3 \leqq a_1(1-a_1^2)+\frac{(4a_1^2-\mu_1)^2}{8a_1}+a_2\frac{2a_2-\mu_1}{2a_1},$$

where μ_1 is the smaller root of

$$(a_2/a_1)+2a_1 = -(\mu_1/2a_1)\log(\mu_1/4a_1e).$$

The defining relations for the extremum functions are given in both cases. The author does not mention in his historical notes that theorem 1 was proved by similar variational methods by W. Janowski [Ann. Polon. Math. **2** (1955), 145–160; MR **17**, 598]. He also displays the one-parameter family of extremum functions in the case $a_1=e^{-1}$ on page 155 of the paper cited above.

G. Springer (Lawrence, Kans.)

110:

Komatu, Yûsaku. On coefficient problems for some particular classes of analytic functions. Kōdai Math. Sem. Rep. **11** (1959), 124–130.

Let $\mathfrak{S}$ be the class of functions $f(z)=z+\sum_{\nu=2}^{\infty} a_\nu z^\nu$ regular and univalent in $|z|<1$ and starlike with respect to the origin. Hummel [Proc. Amer. Math. Soc. **9** (1958), 82–87; MR **20** #1779] proved the following: Let $F(a_2, a_3, \cdots, a_n)$ be any function of the complex variables a_ν having a continuous derivative in each variable; then any function $f(z)$ that maximizes $\Re F(a_2, \cdots, a_n)$ within the class $\mathfrak{S}$ must be of the form

$$f^*(z) = z\prod_{\mu=1}^{m}(1-\kappa_\mu z)^{-\sigma_\mu},$$

where $|\kappa_\mu|=1$, $\sigma_\mu>0$ $(\mu=1, 2, \cdots, m)$, $\sum_{\mu=1}^{m}\sigma_\mu=2$ and $m \leqq n-1$.

Hummel's proof uses variational methods. The author mentions a mild objection to Hummel's method, and then proceeds to give an alternate and simpler proof of the same theorem based on Carathéodory's work on functions with positive real part in the unit circle.

The author also proves two similar theorems: (1) For the class of functions $\phi(z)=z^{-1}+\sum_{\nu=0}^{\infty} a_\nu z^\nu$ which are regular and univalent in $0<|z|<1$ and map that domain onto a domain whose complement is starlike with respect to the origin; and (2) for the subclass of $\mathfrak{S}$ for which the image domain is convex.

A. W. Goodman (Lexington, Ky.)

111:

Erdős, Paul. A remark on the iteration of entire functions. Riveon Lematematika **13** (1959), 13–16.

A theorem on the order of the nth iterates of an entire function f is stated. This is an extension of a result of I. N. Baker [Math. Z. **69** (1958), 121–163; MR **20** #4000]. The outline of the proof given in the article is extremely sketchy and not convincing.

W. J. Thron (Boulder, Colo.)

112:

Bishop, Errett. Approximation by a polynomial and its derivatives on certain closed sets. Proc. Amer. Math. Soc. **9** (1958), 946–953.

Let C be a Jordan arc in the plane and let $f_0, f_1, \cdots, f_n$ be continuous functions on C. The author asks under what conditions there exists a sequence of polynomials $\{p_k\}$ such that $p_k \to f_0$ uniformly on C, $p_k' \to f_1, \cdots, p_k^{(n)} \to f_n$, where superscripts denote derivatives. If C is rectifiable, or has rectifiable subarcs, it is clear that relations between the f_i will have to hold. The author conjectures that if C has no rectifiable subarcs, an approximating sequence exists for every choice of the f_i. He proves this conjecture under an additional hypothesis. Let Y be a homeomorphic map of $[0, 1]$ into the complex plane. Let C denote the Jordan arc which is the image of this map. Say that C satisfies a Lipschitz condition of order c at a point $Y(t)$ of C (t in $[0, 1]$) if there exist $A>0$ and $\beta>0$ such that $\max\{d(Y[0,t], z), d(Y[t,1], z)\} \geq A|Y(t)-z|^c$ whenever $|Y(t)-z|<\beta$ (where for any set S we mean by $d(S, z)$ the distance from S to z). Theorem: If C has no rectifiable subarcs and if there exists $c>0$ such that C satisfies a Lipschitz condition of order c at a dense set of points, then for any continuous functions $f_0, \cdots, f_n$ on C there exists a sequence $\{p_i\}$ of polynomials for which $p_i^{(k)} \to f_k$ uniformly on C as $i \to \infty$, for $0 \leq k \leq n$. The author also shows that if C is a Jordan arc with no rectifiable subarcs and which has a tangent at a dense set of points, then C fulfills the hypotheses of the theorem, and he constructs such arcs.

J. Wermer (Providence, R.I.)

113:

Bishop, Errett. Simultaneous approximation by a polynomial and its derivatives. Proc. Amer. Math. Soc. **10** (1959), 741–743.

The author proves the following result: Let C be a compact set without interior and let E be a compact, totally disconnected subset of C. Let f_0 be a continuous function on C and let $f_1, \cdots, f_n$ be continuous functions on E. Then there exists a sequence of polynomials $\{p_i\}$ such that $p_i \to f_0$ uniformly on C as $i \to \infty$ and $p_i^{(k)} \to f_k$ uniformly on E as $i \to \infty$, for $1 \leq k \leq n$, where $p_i^{(k)}$ is the kth derivative of p_i. The author had previously considered a similar problem for the case when C is a Jordan arc [see preceding review].

J. Wermer (Providence, R.I.)

114:

Erdős, Paul; Piranian, George. Sequences of linear fractional transformations. Michigan Math. J. **6** (1959), 205–209.

A point set E in the extended complex plane is a set of divergence (SD) provided there exists a sequence of linear fractional transformations $T_n(z) = (a_n z + b_n)/(c_n z + d_n)$ which diverges if z is in E and converges if z is in the complement of E. The authors show that a subset of a straight line is an SD if and only if it is of type $G_{\delta\sigma}$. They characterize the SD sets which are countable and show that each subset of a countable SD set is an SD set.

H. S. Wall (Austin, Tex.)

SPECIAL FUNCTIONS

115:

Gould, H. W. Dixon's series expressed as a convolution. Nordisk Mat. Tidskr. **7** (1959), 73–76, 96.

Put

$$(1) \qquad S = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{(n!)^3}.$$

The author shows that

$$(2) \qquad \sum_{k=0}^{n} A_k A_{n-k} = (-1)^n 2^{2n} \binom{2n}{n} S,$$

where

$$A_k = \binom{n+k}{n}\binom{2n+2k}{n+k}.$$

The proof of (2) is accomplished by making use of two different expressions for the Legendre polynomial $P_n(x)$, namely

$$P_n(x) = \frac{1}{2^n} \sum_{2k \leq n} (-1)^k \binom{n}{k}\binom{2n-2k}{n} x^{n-2k}$$

$$= \left(\frac{x-1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k.$$

It is observed that

$$A_k = (-1)^k 2^{2k} \binom{2n}{k}\binom{-n-\frac{1}{2}}{k};$$

substituting this value in (2), (1) follows immediately.

L. Carlitz (Durham, N.C.)

116:

Kelisky, Richard. The numbers generated by exp (arctan x). Duke Math. J. **26** (1959), 569–581.

The integers T_n of the title are defined by means of $\sum_{n=0}^{\infty} T_n x^n/n! = \exp(\arctan x)$ and satisfy the recurrence relation $T_{n+1} = T_n - n(n-1)T_{n-1}$ with $T_0 = T_1 = 1$. In this paper there are derived many interesting arithmetical properties of the T_n. The method makes use of results concerning the difference equation $u_{n+1} = f(n)u_n + g(n)u_{n-1}$, where $f(n)$ and $g(n)$ are polynomials with integral coefficients and $g(0)=0$. In § 2 the author gives four corollaries which follow from a general theorem due to Carlitz [Math. Z. **59** (1954), 474–483; MR **15**, 604] concerning this difference equation. In § 3 he obtains several identities involving the T_n, one of which also involves the numbers of Fibonacci and Lucas. In § 4 he considers some of the arithmetic properties of the T_n and deduces several congruences involving these numbers. Two representative results are the following. (1) If p is an odd prime, then $T_p \equiv 0$ or $2 \pmod p$ according as $p = 4n+1$ or $4n+3$. (2) If p is a prime of the form $4n+1$, $m \geq 1$, $[m/p] = r$, then $T_m \equiv 0 \pmod{p^r}$. In § 5 he obtains expressions for a second solution of the difference equation satisfied by T_n and examines briefly some of the arithmetic properties of the second solution.

A. L. Whiteman (Princeton, N.J.)

117:

Carlitz, L. Some arithmetic properties of a special sequence of polynomials. Duke Math. J. **26** (1959), 583–590.

Generalizing the integers T_n of Kelisky [see the preceding review] the author defines the polynomials $T_n(z)$ by means of the generating function

$$(1) \qquad \left(\frac{1+ix}{1-ix}\right)^{-iz/2} = \exp(z \arctan x) = \sum_{n=0}^{\infty} T_n(z) \frac{x^n}{n!}.$$

It follows from (1) that $T_n(z)$ satisfies the recurrence

(2) $$f_{n+1}(z) = zf_n(z) - n(n-1)f_{n-1}(z),$$

and that $T_n(z)$ has integral coefficients. Theorem 1: If p is an odd rational prime and $a+bi$ is a number of the Gaussian field that is integral (mod p), then

$$T_p(a+bi) \equiv a\{1-(-1)^{(p-1)/2}\} \pmod p.$$

In particular if $p \equiv 1 \pmod 4$, then $T_p(a+bi) \equiv 0 \pmod p$. Theorem 2: For arbitrary $m \geqq 1$, $k \geqq 1$, $n \geqq 0$, $T_n(z)$ satisfies

$$\sum_{s=0}^{k} (-1)^{k-s}\binom{k}{s} T_{n+sm}(z) T_{(k-s)m}(z) \equiv 0 \pmod{k!m^k}.$$

Theorem 3: If $\alpha = a+bi$, p prime $\equiv 1 \pmod 4$ or $\alpha = bi$, p an arbitrary prime, where a, b are rational numbers that are integral (mod p), then $T_n(\alpha) \equiv 0 \pmod{p^r}$, where $r=[n/p]$. The author also constructs a second solution $U_n(z)$ of (2) such that $U_1(z)=0$, $U_2(z)=1$ and derives analogous theorems for the $U_n(z)$.

A. L. Whiteman (Princeton, N.J.)

118:

Barrucand, Pierre. Fonctions elliptiques et transformation de Fourier et de Mellin. C. R. Acad. Sci. Paris **250** (1960), 269–271.

The author finds a number of Mellin-transform pairs involving Jacobian elliptic functions. An example of such a pair is the pair

$$\mathrm{sn}\,(K' \log x/\pi, k)/(x-1), \quad \pi\,\mathrm{sn}\,(2Ks, k)/\sin \pi s,$$

where $K=K(k)$, $K'=K(k')$, and $k^2+k'^2=1$. By suitable changes of variable, corresponding Fourier transform pairs are found.

P. G. Rooney (Toronto)

119:

Alavi, Y.; Wells, C. P. Expansions of parabolic wave functions. Proc. Amer. Math. Soc. **10** (1959), 876–880.

The authors expand a product of parabolic wave functions in a series of products of Bessel and Legendre functions. This corresponds to the representation of a paraboloidal wave as a superposition of spherical waves. The coefficients in the expansion are shown to satisfy a recurrence relation and are thereby determined in terms of Pasternack polynomials. The paper contains also an alternative derivation of the main result, an inverse result representing a spherical wave function as an integral involving products of parabolic wave functions, and some special cases and related results.

A. Erdélyi (Pasadena, Calif.)

120:

Al-Salam, Waleed A. Some functions related to the Bessel polynomials. Duke Math. J. **26** (1959), 519–539.

Continuation of an earlier paper by the same author [same J. **24** (1957), 529–545; MR **19**, 849]. The author now introduces an independent second solution $Z_n^{(a)}(x)$ of the differential equation and an independent second solution $V_n^{(a)}(x)$ of the difference equation satisfied by the Bessel polynomials. The functions $Z_n^{(a)}(x)$ are shown to be expressible in terms of Whittaker functions. A number of formal results, most of them transcriptions of known results concerning Whittaker functions, are stated. Using results of H. S. Wall [*Analytic theory of continued fractions*, Van Nostrand, New York, 1948; MR **10**, 32], the polynomials $V_n^{(a)}(x)$ are shown to be orthogonal on the unit circle with respect to the weight function

$$x[{}_1F_1(1; 2+\alpha; -2x^{-1})]^{-1},$$

and a bound for the moduli of their zeros is given. The Eisenstein criterion is used to derive various conditions for the irreducibility of the polynomial $V_n^{(a)}(x)$ in the field of rational numbers.

P. Henrici (Los Angeles, Calif.)

121:

Babister, A. W. Generalized modified Struve functions. Quart. J. Math. Oxford Ser. (2) **10** (1959), 214–223.

The author introduces the function $\Omega(a, c, x)$ which is a particular solution of the inhomogeneous confluent hypergeometric equation

$$x\Omega'' + (c-x)\Omega' - a\Omega = \frac{2^{1-c}\Gamma(c)}{\Gamma(a)\Gamma(c-a)} e^{x/2},$$

and considers also $\bar{\Omega}(a, c, x) = x^{1-c}\Omega(a-c+1, 2-c, x)$. He obtains various integral representations, recurrence relations, and differentiation formulas; and he relates $\Omega(a, 2a, x)$ to the modified Struve function $L_{a-1/2}(x/2)$. [A somewhat different inhomogeneous confluent hypergeometric equation was investigated by H. Buchholz, Math. Z. **57** (1953), 167–192; MR **14**, 748.]

A. Erdélyi (Pasadena, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 48, B346.

122:

Braier, Alfred. Méthode graphique pour l'étude des systèmes non linéaires dans le plan des phases. Bul. Inst. Politehn. Iași (N.S.) **4** (8) (1958), no. 3–4, 107–112. (1 insert) (Russian and Romanian summaries)

De l'introduction de l'auteur: "Dans cette note on expose une méthode graphique qui permet de construire les trajectoires du plan de phase de même que les solutions des équations différentielles $\ddot{x}+f(x, \dot{x}) = F(t)$."

S. Lefschetz (Mexico City)

123:

Hartman, Philip. On exterior derivatives and solutions of ordinary differential equations. Trans. Amer. Math. Soc. **91** (1959), 277–293.

This paper is concerned with conditions for the local uniqueness of solutions (and for C^1 character of general solutions) of ordinary differential equations, and with applications of the results to Riemannian geometry and other topics. For a differential form (not necessarily differentiable) on a domain, we can introduce a notion of "possessing an exterior derivative" by requiring Stokes' formula for the exterior derivative. The above-mentioned conditions are in terms of exterior derivatives and generalize several results already published by the author. Two main general results are the following: (I) Let

$f(x, s)$ be a continuous n-vector valued function defined for $|x|<1$, $|s|<1$, where x is a n-vector and s is a real parameter. A sufficient condition for the (local) uniqueness of a solution of the initial value problem $dx/ds=f(x, s)$ and $x(0)=0$, is that there exist a continuous n by n matrix function $A=(a_j^i(x, s))$ on $|x|<\varepsilon$, $|s|<\varepsilon$, for some $\varepsilon>0$, such that $\det A\neq 0$ and that n Pfaffians

$$(\omega_j) = (a_j^k dx_k - a_j^k f_k ds) = A(dx-fds)$$

have bounded exterior derivatives on $|x|<\varepsilon$, $|s|<\varepsilon$. (II) Let $f(x, s, z)$ be a continuous n-vector valued function on $|x|<1$, $|s|<1$, $|z|<1$, where z is a m-vector. A necessary and sufficient condition that the initial value problem $dx/ds=f(x, s, z)$ and $x(0)=v$, in which z is a parameter, have a unique solution $x(v, s, z)$ which is of class C^1 in all of its variables for small v, s, z, is that there exist a continuous non-singular n by n matrix valued function $A(x, s, z)$ and a continuous n by m matrix valued function $C(x, s, z)$ defined for $|x|<\varepsilon$, $|s|<\varepsilon$, $|z|<\varepsilon$, for some $\varepsilon>0$, such that n Pfaffian forms $\omega=A(dx-fds)+Cdz$ have continuous exterior derivatives on $|x|<\varepsilon$, $|s|<\varepsilon$, $|z|<\varepsilon$.

The above results are applied to the following: (i) a (local) unique existence theorem for geodesics for a Riemannian metric of class C^1 and possessing a (generalized) bounded curvature tensor, and a condition for the C^1-dependence of geodesics on initial data; (ii) (i) is applied to C^2-surfaces in euclidean spaces; (iii) problems of extremals for more general parametric problems of the calculus of variations; (iv) a theorem of Frobenius for completely integrable systems of continuous Pfaffian forms possessing exterior derivatives. *M. Kuranishi* (Nagoya)

124:

Perčinkova-V'čkova, Danica. Compléments au traité de Kamke. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **10** (1957), 37–42. (Serbo-Croatian. French summary)

On donne quelques cas d'intégrabilité par quadratures de l'équation différentielle

$$xS(x, y)dy+yT(x, y)dx = 0,$$

où $S(x, y)$ et $T(x, y)$ représentent des polynômes à deux variables x et y.

Cette note se rattache à un article de D. S. Mitrinović [Glasnik. Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **11** (1956), 7–10; MR **18**, 214]. *D. S. Mitrinović* (Belgrade)

125:

Dolapčiev, Bl.; Čobanov, Iv. Über eine Differentialgleichung von J. Halm. Bŭlgar. Akad. Nauk Izv. Mat. Inst. **3**, no. 1, 51–68 (1958). (Bulgarian. Russian and German summaries)

Zusammenfassung der Autoren: "Von gewissen hydrodynamischen Tatsachen ausgehend, die mit der Bewegung der idealen Flüssigkeit bei Anwesenheit einer zweiseitig unendlichen, schachbrettartigen oder symmetrischen Wirbelstraße verbunden sind, und die Struktur der Riccatischen Differentialgleichungen berücksichtigend, auf welche diese Flüssigkeitsbewegung zurückführt, werden Integrale der Differentialgleichung von J. Halm

$$(1) \qquad (1+x^2)^2y''+(ax^2+b)y = 0$$

in geschlossener Form, die nur in den Fällen $a=0$ und $a=b+3$ von Halm gelöst ist, auch für andere Beziehungen zwischen den Parametern a und b gesucht. Dadurch wird eine Methode angewandt, die zur Lösung homogener linearer Differentialgleichungen zweiter Ordnung dient, bei gewissen Zusammenhängen zwischen ihren Parametern, wobei von vornherein geeigneten Riccatischen Differentialgleichungen mit bekannten Integralen konstruiert werden, auf welche die Transformation zu den entsprechenden homogenen linearen Differentialgleichungen zweiter Ordnung angewendet wird.

"Diese Methode ist auch durch das Analogon

$$(2) \qquad (1-x^2)^2y''+(ax^2+b)y = 0$$

der Halmschen Differentialgleichung (1) illustriert, indem Anwendungen auf verwandte Differentialgleichungen, die durch geeigneten Transformationen durch (1) und (2) zurückführen, gemacht werden.

"Es werden neue Fälle erhalten, bei denen diese verwandten Differentialgleichungen auch in geschlossener Form gelöst werden, und manche Verallgemeinerungen gegeben."

126:

Hukuhara, Masuo; Iwano, Masahiro. Étude de la convergence des solutions formelles d'un système différentiel ordinaire linéaire. Funkcial. Ekvac. **2** (1959), 1–18. (Esperanto summary)

The authors consider the vector differential equation

$$(*) \qquad \frac{dy}{dx} = A(x)y$$

in the n-dimensional vector function $y(x)$, where the elements of the $n\times n$ coefficient matrix $A(x)$ are meromorphic at $x=0$. Employing the same general methods by which Iwano [J. Fac. Sci. Univ. Tokyo. Sect. I **7** (1956), 343–351; MR **18**, 210] obtained the results of Perron on the number of linearly independent solutions of an nth order linear differential equation which are regular at $x=0$, the authors determine the number of linearly independent solutions of (*) expressible as

$$y(x) = e^{\Lambda_h(x)}x^{\rho_h}\sum_{j=0}^{h}\frac{(\log x)^{k-j}}{(k-j)!}\,\mathfrak{y}_j(x),$$

in which the ρ_h are constants, $\Lambda_h(x)=\int_\infty^x \lambda_h(x)dx$, with $\lambda_h(x)$ a polynomial $\lambda_h/x^{\sigma_h+1}+\cdots+\omega_h/x^{1/q_h+1}$ in a fractional power x^{-1/q_h}, and the $\mathfrak{y}_j(x)$ are holomorphic functions of x^{1/q_h} at $x=0$. The general results of the paper are applied to an nth order equation

$$x^s y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_n(x)y = 0,$$

where the $a_1(x), \cdots, a_n(x)$ are holomorphic at $x=0$. *W. T. Reid* (Iowa City, Iowa)

127:

Yo, Ming-gen. On stability and boundedness of solutions of non-linear differential equations. Advancement in Math. **3** (1957), 209–215. (Chinese. English summary)

128:

Hahn, Wolfgang. Bemerkungen zu einer Arbeit von Herrn Vejvoda über Stabilitätsfragen. Math. Nachr. **20** (1959), 21–24.

O. Vejvoda [Časopis Pěst. Mat. **82** (1957), 137–159; MR **20** #1044] has discussed the stability of solutions of systems of differential equations in the complex domain by using the direct method of Lyapunov. The main purpose of the paper under review is to show that the main tools needed by Vejvoda can be derived from known results in the real domain. This is done by separating the equation into its real and imaginary parts.

J. K. Hale (Baltimore, Md.)

129:

Chin, Yuan-shun, et al. Concrete examples of existence of three limit cycles for the system $dx/dt=X_2(x, y)$, $dy/dt=Y_2(x, y)$. Acta Math. Sinica **9** (1959), 213–226. (Chinese. English summary)

Let $P(x, y)$ and $Q(x, y)$ be polynomials of second degree. N. N. Bautin proved that some differential systems of the type (*) $dx/dt = P$, $dy/dt=Q$ can have three limit cycles [Mat. Sb. (N.S.) **30 (72)** (1952), 181–196; MR **13**, 652], but I. G. Petrovskiĭ and E. M. Landis established that the system (*) can have at most three [Mat. Sb. (N.S.) **37 (79)** (1955), 209–250; MR **17**, 364]. In this paper the authors, combining the theory of Bautin and a method due to M. Frommer [Math. Ann. **109** (1934), 395–424], develop a method of constructing concrete examples of differential systems of the type (*) with three limit cycles. One such concrete differential system is explicitly given which, as indicated by the authors, is the first such example to appear in the literature.

Choy-tak Taam (Washington, D.C.)

130:

Bellman, Richard; Cooke, Kenneth L. On the limit of solutions of differential-difference equations as the retardation approaches zero. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1026–1028.

A theorem is stated to the effect that a first order nonhomogeneous linear difference-differential equation of mixed difference type has a solution which, as its lag parameter approaches zero, tends toward that of the corresponding differential equation, subject to a few mild conditions.

E. Pinney (Berkeley, Calif.)

131:

Van, Lian'. Stability of solutions of equations with lagging argument. Sci. Record (N.S.) **3** (1959), 280–288. (Russian)

Some results are presented concerning the asymptotic behavior of solutions of vector equations of the form $dx/dt=P(t)x(t)+Q(t)x(t-g(t))+h(x(t), x(t-g(t)))$, under various assumptions concerning the scalar function $g(t)$ and the vector function h.

R. E. Bellman (Santa Monica, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also B450.

132:

Cordes, H. O. On continuation of boundary values for partial differential operators. Pacific J. Math. **9** (1959), 987–1011.

Soit $L=\sum_{i=1}^{n} a_i(x)\partial/\partial x_i+b(x)$ un opérateur différentiel du premier ordre agissant sur les fonctions vectorielles, à valeurs dans C^m, définies sur un domaine borné D de R^n, de frontière Γ régulière. Pour la résolution des problèmes aux limites relatifs à L, il est important de savoir si une fonction vectorielle u_0 définie sur Γ admet un prolongement u dans D tel que Lu soit mesurable et de carré sommable dans D. Posant $A=\sum a_i\nu_i$ (où $\{\nu_i\}$ désigne la normale à Γ) l'auteur montre que ce prolongement est possible (moyennant certaines précisions) si Au_0 satisfait à une condition de Lipschitz sur Γ.

J. Lelong (Paris)

133:

Haimovici, Adolf. Sur l'immersion de l'espace des solutions d'une équation aux dérivées partielles linéaire du second ordre, dans l'espace des solutions d'une équation du même type, à un nombre plus grand de variables. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 43–48.

Using terminology and methods of Riemannian geometry, the author discusses the conditions under which the second order linear homogeneous partial differential equation can, by means of a procedure involving an increase in the number of independent variables, followed by a non-singular transformation of the independent variables, be put into one of various special forms in each of which the second order part is the Laplacian.

W. Strodt (New York)

134:

Moisil, Gr. C. Formules de réciprocité et systèmes différentiels adjoints. V. Com. Acad. R. P. Romîne **9** (1959), 229–231. (Romanian. Russian and French summaries)

[For part IV, see Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4** (1952), 39–51; MR **15**, 960.]

For systems of differential equations of hypercomplex variables, the author gives reciprocity and adjoint system formulas. These are illustrated for Maxwell's equations.

J. A. Ward (Alamogordo, N.M.)

135:

Lebedev, V. I. The method of nets in the second boundary value problem for Poisson's equation. Dokl. Akad. Nauk SSSR **127** (1959), 742–745. (Russian)

The author exhibits a method for the approximate solution of the two-dimensional Neumann problem for the Poisson equation $\Delta u=-\operatorname{div}(f_1, f_2)$. The method consists of formulating a finite difference problem for the conjugate function v defined by the transformation

$$\partial u/\partial x_1+f_1 = \partial v/\partial x_2, \quad \partial u/\partial x_2+f_2 = -\partial v/\partial x_1.$$

The conjugate function satisfies a Dirichlet condition around the boundary just as it does with Laplace's equation. This simplifies the formulation of the finite difference problem for v. In the final transition from v to u, both linear and quadratic interpolation of the boundary values of v are used.

A. C. Downing (Oak Ridge, Tenn.)

136:

Odhnoff, Jan. Operators generated by differential problems with eigenvalue parameter in equation and boundary condition. Medd. Lunds Univ. Mat. Sem. 14 (1959), 79 pp; corrections, unbound insert.

The author discusses the operator $\mathcal{A}=-[\partial/\partial n, \Delta]$ in the space $L^2=L^2(\Gamma)\oplus L^2(\Omega)$, where Δ is the Beltrami operator in a Riemannian manifold V^m, Ω is an open subset of V^m with smooth boundary S, and Γ is an open part of S. The closure A_0 of $\mathcal{A}$ in L^2 is studied and the spectral theory of an arbitrary self-adjoint extension A of A_0 is treated. A generalized eigenfunction expansion for A is found, and, when A is semi-bounded, the existence of a spectral function e_λ is proved. A kernel $g_t^{(p)}$ of the operator $(A+t)^{-p}$ is shown to exist when $-t$ does not belong to the spectrum of A. The asymptotic behavior of e_λ is investigated when A is semi-bounded. Lastly, conditions on A are given in order that $H=AP$ be self-adjoint in $L^2(\Gamma)$, where P is the operator in $L^2(\Gamma)$ such that Pu is defined in $\bar{\Omega}$, $\Delta(Pu)=0$ in Ω, $Pu=u$ on Γ, and such that Pu satisfies some homogeneous boundary conditions on $S-\Gamma$ (e.g., $Pu=0$ or $\partial Pu/\partial n=0$). These considerations stem from the eigenvalue problem $\Delta u=0$ in Ω, $\partial u/\partial n+\lambda u=0$ on Γ, with u satisfying the prescribed boundary values on $S-\Gamma$. The latter problem was first considered by Pleijel [Proc. Sympos. Spectral Theory Differential Problems, pp. 439–454, Okla. Agri. and Mech. College, Stillwater, Okla., 1951; MR 13, 948].

M. Schechter (New York)

137:

Padmavally, K. A Poincaré problem. J. Indian Math. Soc. (N.S.) 22 (1958), 181–205 (1959).

A boundary value problem connected with the neutron flux in a reactor in the steady state involves the determination of $\varphi(x, y)$ satisfying $\nabla^2\varphi(x, y)=\varphi(x, y)-1$, subject to the conditions $\varphi(x, y)=0$ on S: $[-1\leq x\leq 1, 0]$ and $[0, -1\leq y\leq 1]$ and $\varphi(x, y)\to 1$ as $x^2+y^2\to\infty$. It is shown that this problem has a unique continuous solution, by setting up an equivalent boundary value problem, showing that the solution of this is reducible to that of an integral equation of the form $\int_0^1 \theta(x)K(x, 0; \xi, 0)dx=-2\pi$ for a $\theta(x)$ satisfying certain conditions, $K(x, y; \xi, \eta)$ being expressible in terms of Hankel functions of order zero, and showing that this in turn is equivalent to the solution of a Fredholm equation of the second kind.

T. H. Hildebrandt (Ann Arbor, Mich.)

138:

Brownell, F. H. A note on Kato's uniqueness criterion for Schrödinger operator self-adjoint extensions. Pacific J. Math. 9 (1959), 953–973.

Extending results of Kato [Trans. Amer. Math. Soc. 70 (1951), 195–211; MR 12, 781], it is proved that if $V(x)$ is essentially bounded over R^n, and $V(x)^{(n+\rho)/2}$ is summable over R^n for some positive ρ such that $n+\rho\geqq 2$, $n+\rho=4$ if $n<4$, then the operator $-\nabla^2u+Vu$ has a unique selfadjoint extension in the Hilbert space $L(R^n)$. It is also shown that the set of eigenvalues, if they exist, may be defined by minimax conditions such as hold for compact operators.

J. L. B. Cooper (Cardiff)

139:

Adler, G.; Freud, G. Una applicazione del calcolo degli operatori di Mikusiński per la risoluzione d'una equazione alle derivate parziali. Magyar Tud. Akad. Mat. Kutató Int. Közl. 4 (1959), 367–375. (Hungarian and Russian summaries)

The authors use Mikusiński's operational calculus to solve the following boundary value problem for $Z(x, t)$: $-Z_{xx}+x^{-1}Z_x-Z_t-x^{-2}Z=0$ for $x>1$, $t>0$; $Z(x, 0)=0$ for $x>1$; $Z(1, t)=g(t)$ for $t>0$. They first find the solution, z, in the particular case $g(t)=1$ for all t, and then, for a piecewise continuously differentiable function g, in the form

$$Z(x, t) = \frac{\partial}{\partial t}\int_0^t g(u)z(x, t-u)du.$$

{Referee's remarks: (i) This is the form of the solution given in the Hungarian and Russian summaries. In the text, $Z(x, t)=g(0)z(x, t)+g' * z(x, t)$, where $*$ denotes convolution; and it appears that for this latter form g must be continuous. (ii) The authors stress that at a certain stage of their work a generalized function appears. This is due to their procedure rather than the nature of the problem which could have been solved by routine Laplace transform techniques.}

A. Erdélyi (Pasadena, Calif.)

140:

Friedman, Avner. On quasi-linear parabolic equations of the second order. J. Math. Mech. 7 (1958), 793–809.

Let D be a finite domain in $(n+1)$-dimensional (x, t)-space and S a $(2n+2)$-dimensional cylinder over $\bar{D}$ in (x, t, u, v)-space, where $x=(x_1, \cdots, x_n)$ and $v=(v_1, \cdots, v_n)$. D is supposed to be bounded by two planes, $t=0$ and $t=t_1>0$, and a lateral surface C between them; ∂D denotes the part of the boundary of D not in the plane $t=t_1$. A neighborhood of each point of C is supposed to be representable for some index i $(1\leq i\leq n)$ as

$$x_i = h(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t),$$

where h is sufficiently smooth. Let

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i(x, t)\frac{\partial u}{\partial x_i}+c(x, t)u-\frac{\partial u}{\partial t}$$

be a linear, parabolic, second order differential expression with sufficiently smooth coefficients defined in $\bar{D}$, and let $f(x, t, u, v)$ be a function continuous in S. Existence and uniqueness are considered for the solution in D of the equation $Lu=f(x, t, u, \nabla u)$ $(\nabla u=(\partial u/\partial x_1, \cdots, \partial u/\partial x_n))$ which vanishes on ∂D, and analogous questions for second order equations of elliptic type are also discussed. The results as to uniqueness presume $c(x, t)\leq 0$ and some kind of monotonicity of f and are based on comparison theorems. To be assured of existence, f is required to be Hölder-continuous in any compact subset of S and also to satisfy a certain further condition such as

$$|f(x, t, u, v)| \leq A+B|u|^\lambda+C|v|^\mu$$

with positive constants A, B, C, λ, μ. These constants are subject to additional restrictions, such as $\lambda<1$, $\mu<1$, unless t_1 is sufficiently small. The proof of existence is based on considering a transformation $w=Tu$, applied to functions u such that u and ∇u are Hölder-continuous in $\bar{D}$, where w is the solution in D of the linear equation $Lw=f(x, t, u, \nabla u)$ which vanishes on ∂D. A sphere in the

space of such functions u, properly normed, is shown to be mapped by T into itself, and Schauder's fixed point theorem yields one form of the desired statement as to existence. The results generalize previous work of C. Ciliberto, T. Sato, and B. Pini.

A. Douglis (College Park, Md.)

141:

Bhutani, O. P. A certain boundary value problem and its applications. Appl. Sci. Res. A 8 (1959), 413–424.

Als Ausgangspunkt dient die partielle Differentialgleichung der Diffusion in elliptischen Zylinderkoordinaten. Als Lösung wird eine Laplacetransformation angesetzt, wobei die transformierte Funktion der genannten Gleichung, unter Berücksichtigung der Randbedingungen in bezug auf die Zeit, genügt. Für sie gilt eine unendliche Reihenentwicklung nach Produkten von Mathieuschen Funktionen und zugeordneten Mathieuschen Funktionen. Mit Hilfe des Mellinschen Inversionssatzes ergibt sich hieraus die Unterfunktion. Für die Mathieuschen Funktionen werden Reihendarstellungen nach Besselschen Funktionen benutzt. Die Lösung wird für spezielle Formen der Randbedingungen betrachtet. Als Anwendungen werden behandelt: die Bewegung einer inkompressiblen viskosen Flüssigkeit in einem Rohr von elliptischem Querschnitt, die Wärmekonvektion in einem solchen Zylinder und die Temperaturbestimmung darin, sowie die Wirbelstromverluste in einem elliptischen Zylinder.

M. J. O. Strutt (Zürich)

142:

Rykalin, N. N. On conditions for splitting the solutions of linear parabolic equations into orthogonal components. Dokl. Akad. Nauk SSSR **125** (1959), 519–522 (Russian); translated as Soviet Physics. Dokl. 4, 293–297.

Let $u(P_n, t)$ be a solution of the linear parabolic differential equation

$$(1)\qquad \frac{\partial u}{\partial t} = \sum_{i=1}^{n}\left(a_{ni}\frac{\partial^2 u}{\partial x_i^2}+b_{ni}\frac{\partial u}{\partial x_i}\right)+c_n u,$$

where $P_n=(x_1, \cdots, x_n)$ belongs to a region V in n-dimensional space either infinite or bounded by a hypersurface S_n with the condition

$$(2)\qquad \frac{\partial u}{\partial n}+f_n u = 0 \quad (P_n\in S_n,\ t\in(-\infty, \infty)).$$

The coefficients of (1) may depend both on P_n and t, and at least one of the a_{ni} is different from zero everywhere in V. The author finds conditions under which $u(P_n, t)$ can be expressed as a product

$$(3)\qquad u(P_n, t) = u_k(P_k, t)\cdot u_m(P_m, t),$$

where $k+m=n$, $P_k=(x_1, \cdots, x_k)$, $P_m=(x_{k+1}, \cdots, x_n)$, with u_k and u_m satisfying equations and boundary conditions of the same type as u, but in lower dimensions.

J. Elliott (New York)

143:

Cordes, H. O. Vereinfachter Beweis der Existenz einer Apriori-Hölderkonstanten. Math. Ann. **138** (1959), 155–178.

The author presents a simplified—though still intricate—proof of his important result concerning the Dirichlet problem for linear elliptic differential equations in n variables [Math. Ann. **131** (1956), 278–312; MR **19**, 961],

$$(*)\quad \begin{cases} L(u) \equiv \sum_{i,k=1}^{n} a_{ik}(x)u_{ik}+\sum_{i=1}^{n} b_i(x)u_i+b_0(x)u-f(x)=0 & \text{for } x\in B,\\ u=\varphi(x) \quad \text{for} \quad x\in\Gamma = \text{boundary of } B, \end{cases}$$

based on sharp estimates for square integrals of derivatives of u, similar to the ones used by C. B. Morrey [Trans. Amer. Math. Soc. **43** (1938), 126–166], L. Nirenberg [Comm. Pure Appl. Math. **6** (1953), 103–156; addendum, 395; MR **16**, 367], K. O. Friedrichs, E. Heinz [Nachr. Akad. Wiss. Göttingen. IIa **1955**, 1–12; MR **17**, 626], and the author. It has immediate applications to the Dirichlet problem for quasi-linear equations.

Assumptions: B is a bounded domain in E^n whose boundary Γ consists of finitely many C^2-hypersurfaces. The coefficients $a_{ik}(x)$ are measurable and satisfy $p\le\sum a_{ii}(x)\le P$ in B. For any $x^0\in B$ there exist a sphere S_{x^0}: $|x-x^0|\le\tau$ and a matrix $T(x^0)$ with $q|\xi|\le|T(x^0)\xi|\le Q|\xi|$, where $0<q\le Q$, such that the matrix

$$T'(x^0)(a_{ik}(x))T(x^0)$$

satisfies a K_ε-condition (definition below) in $B\cap S_{x^0}$. Here q, Q, ε, τ are independent of x^0 and x. The functions $b_i(x)$ are bounded and measurable in B: $\sum_{i=0}^{n} b_i^2\le M$.

Let $u(x)$ be a $C^1(B)$-solution of (*) with piecewise continuous second derivatives. Then there exist an exponent $\alpha>0$ and constants c_1, c_2, c_3, the latter depending on n, p, P, q, Q, M, ε, τ, α only, such that the Hölder-coefficient $H_\alpha(u; B)$ satisfies the apriori inequality

$$H_\alpha(u; B) \le c_1\|f\|+c_2\|u\|+c_3\|\varphi\|_2^1.$$

A positive definite matrix (a_{ik}) is said to satisfy a K_ε-condition (cone condition) if $n(n-1)\sum a_{ik}^2\le(n-\varepsilon)\times(\sum a_{ii})^2$. It has the following geometric interpretation. Associate with (a_{ik}) the point $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ of a λ_i-space, where the λ_i are the eigenvalues (arranged in some order). Let K_0 be the cone with axis $\lambda_1=\lambda_2=\cdots=\lambda_n$, touching the hyperplanes $\lambda_i=0$. Its equation is $(n-1)\sum_{i<k}(\lambda_i-\lambda_k)^2\le(\sum\lambda_i)^2$. Then the point λ is required to be in a slightly smaller cone K_ε: $(n-1)\sum_{i<k}(\lambda_i-\lambda_k)^2\le(1-\varepsilon)(\sum\lambda_i)^2$. The condition of ellipticity would mean $\lambda_i>0$.

J. C. C. Nitsche (Minneapolis, Minn.)

144:

Schmidt, Wolfgang. Einige Sätze über die Riemann-Hilbertsche Randwertaufgabe. Wiss. Z. Hochschule Elektrotechn. Ilmenau **3** (1957), 9–12.

The author extends recent results of I. N. Vekua concerning elliptic systems of the form

$$u_x-v_y = au+bv+f,$$
$$u_y+v_x = cu+dv+g$$

with boundary condition

$$\alpha u+\beta v = \gamma,$$

where the real-valued functions a, b, c, d, f, g, α, β, γ satisfy suitable restrictions; in particular, α and β must be Hölder-continuous and not simultaneously zero. This condition is relaxed in the present paper to allow $\alpha+i\beta$ to have a finite number of poles and zeroes of fractional

order (i.e., <1). As in Vekua's work, the essential feature is the conversion of the boundary-value problem to a singular integral equation.

Bernard Epstein (Philadelphia, Pa.)

145:

Schmidt, W. Über eine verallgemeinerte Riemann-Hilbertsche Randwertaufgabe. Wiss. Z. Hochschule Elektrotechn. Ilmenau **4** (1958), 25–30.

The author extends recent results of J. Nitsche concerning solutions of systems of the form

$$u_x - v_y = au + bv,$$
$$u_y + v_x = cu + dv$$

with linear boundary conditions, either homogeneous or non-homogeneous. Among other results, Nitsche obtained for a (non-trivial) solution of the homogeneous problem the inequality $2N_T + N_S \leq 2n$, where N_T and N_S denote the number of zeroes of $u+iv$ in the domain and on the boundary, respectively, while n denotes the index. In the present paper the author modifies Nitsche's methods, which were restricted to simply-connected domains, so as to apply to domains of arbitrary finite connectivity.

Bernard Epstein (Philadelphia, Pa.)

146:

Yamaguti, Masaya. Le problème de Cauchy et les opérateurs d'intégrale singulière. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 121–151.

The author finds criteria for the "proper solution" of the Cauchy problem for a hyperbolic partial differential equation with variable coefficients in n dimensions ($n \geq 2$) by generalizing the methods of Calderón and Zygmund [Amer. J. Math. **79** (1957), 901–921; **80** (1958), 16–36; MR **20** #7196; **21** #3675]. This extends a result of A. Lax for the case $n=2$ [Comm. Pure Appl. Math. **9** (1956), 135–169; MR **18**, 397].

J. Elliott (New York)

147:

Mizel, V. J. A boundary layer result for an n-dimensional linear elliptic equation. Proc. Amer. Math. Soc. **10** (1959), 775–783.

The boundary value problem

$$(1)\qquad L_\varepsilon u \equiv \varepsilon \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + c(x) = d(x) \text{ in } D,$$
$$u = \bar{u}(x) \text{ on } \dot{D}$$

is studied asymptotically for small $\varepsilon > 0$. The operator L_ε is assumed to be uniformly elliptic in the bounded domain D of E^n; $x = (x_1, x_2, \cdots, x_n)$; $\dot{D}$ is the boundary of D; $c(x) < 0$ in $D \cup \dot{D}$, and $\bar{u}(x)$ is a prescribed function. The result is a generalization from two to n dimensions of a theorem by N. Levinson [Ann. of Math. (2) **51** (1950), 428–445; MR **11**, 439]. It can be roughly summarized as follows: In every "tubeshaped" subdomain of D that is bounded laterally by characteristics of the reduced equation

$$(2)\qquad L_0 u = d(x)$$

and by portions of $\dot{D}$ at both ends, the solution of problem (1) is—to within terms of order $O(\varepsilon^{1/2})$—equal to the sum $U(x) + z(x, \varepsilon)$, where $U(x)$ is the solution of (2) determined by the boundary values $\bar{u}(x)$ at the points where the characteristics of (2) enter the tube, while $z(x, \varepsilon)$ is of the form $h(x)e^{-g(x)/\varepsilon}$. Here $g(x) = 0$ on the end of the tube where the characteristics of (1) leave D, and $g(x) > 0$ in the tube. Thus $z(x, \varepsilon)$ is of a form that has frequently been called a "boundary layer" term.

The proof is based on the maximum principle and makes use of a simplification of Levinson's approach introduced by S. L. Kamenomostskaya [Mat. Sb. (N.S.) **31** (**73**) (1952), 703–708; MR **14**, 877].

W. Wasow (Madison, Wis.)

148:

Wloka, Josef. Über die Anwendung der Operatorenrechnung auf lineare Differential-Differenzengleichungen mit konstanten Koeffizienten. J. Reine Angew. Math. **202** (1959), 107–128.

The author employs operational procedures to investigate the theory of linear difference-differential equations. For equations of mixed differences his results are standard. However, a novel feature of his work is the treatment of partial differential equations with derivatives with respect to two independent variables and differences with respect to one of them.

E. Pinney (Berkeley, Calif.)

POTENTIAL THEORY

149:

Shapiro, Victor L. Intrinsic operators in three-space. Pacific J. Math. **9** (1959), 1257–1267.

Continuing work by himself and also by others, the author establishes the following theorem and related ones. In a domain D of E_3 a continuous vectorfield v is locally in D the gradient of a potential of a continuous density distribution if and only if the "intrinsic" curl is zero and the "intrinsic" divergence of v is continuous. Herein, "intrinsic" means computable point-by-point, without uniformity requirements, and without the use of derivatives, but as the limit of certain integrals over suitable small figures around the point. For the curl this can be done in two alternate ways, either by using small spheres, or by using small circles in suitable planes.

S. Bochner (Princeton, N.J.)

150:

Hadamard, J. Sur le théorème de A. Harnack. Publ. Inst. Statist. Univ. Paris **6** (1957), 177–181.

Sia

$$Au = \sum_{i,k}^{1,n} a_{ik}\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i}^{1,n} b_i \frac{\partial u}{\partial x_i} + cu \qquad (c \leq 0)$$

un operatore ellittico in un dominio Ω connesso. L'A. dà una nuova dimostrazione del seguente teorema (estensione di un noto teorema di Harnach): una serie di soluzioni della equazione $Au=0$, a termini tutti positivi in Ω, convergente in un punto interno di Ω, converge uniformemente in ogni dominio Ω' interno a Ω e può essere derivata termine a termine. La dimostrazione è ottenuta rapidamente, utilizzando la risoluzione del problema di Dirichlet, per una sfera sufficientemente piccola, col

metodo classico della rappresentazione mediante potenziali "generalizzati" di doppio strato,

$$\int_S \frac{ds(x, y)}{d\nu_y} m(y) dS_y$$

(dove $s(x, y)$ è una opportuna soluzione fondamentale di $Au=0$, $d/d\nu_y$ è la cosidetta derivata "conormale" associata ad A) e delle relative equazioni integrali. Il metodo comporta però che i coefficienti di A siano sufficientemente regolari in Ω (per il caso di ipotesi assai più generali sui coefficienti si veda l'articolo di J. Serrin [J. Analyse Math. **4** (1955/56), 292–308; MR **18**, 398] e le ulteriori indicazioni bibliografiche ivi date).

E. Magenes (Pavia)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 35.

151:

Meschkowski, Herbert. ★Differenzengleichungen. Studia Mathematica, Bd. XIV. Vandenhoeck & Ruprecht, Göttingen, 1959. 243 pp.

The central theme of this book is a discussion of various methods for demonstrating the existence and properties of solutions of linear ordinary difference equations. The topics treated in this connection include Laplace integrals over various contours, factorial series, Newton series, generating functions, various iterative procedures, Hurwitz's method for finding an entire solution of $y(x+1)-y(x)=\phi(x)$ if ϕ is entire, the operator methods of Boole and Milne-Thomson for linear equations with rational coefficients, and the Poincaré-Perron theorems on $\lim f(x+1)/f(x)$ (as $x\to\infty$) for solutions of difference equations with asymptotically constant coefficients. Some attention is given to problems where the independent variable assumes only integral values, but most of the book is devoted to problems where the independent variable is a complex variable. Special consideration is given to the Bernoulli polynomials and to the Γ function and related functions.

Several chapters are devoted to 'geometric difference' equations (otherwise known as 'q-difference' equations). It is shown that the solutions of $u(qx)-au(x)=b(x-1)^{-1}$ satisfy no algebraic differential equation; the proof is a variant of Hölder's original proof of the theorem that $\Gamma'(x)/\Gamma(x)$ satisfies no algebraic differential equation.

To any one who has had a first course in real and complex analysis, the book offers a rapid and pleasant introduction to a substantial part of the varied literature on the analytic theory of difference equations. The topics chosen are important and clearly presented. There are instructive exercises on most chapters, and a short but useful bibliography.

The reviewer was disappointed to find no mention of the methods of Guichard and of Sheffer for local solutions. Another possible criticism is that there is perhaps a deficiency of motivation and unification of the various methods. (In this connection it is noteworthy that the author barely mentions the concept of 'principal solution', a concept which appears to the reviewer to provide the best available point of view from which to introduce motivation and unification into the rather incoherent literature of difference equations.)

The convergence proofs are conducted with care. The reviewer, however, questions one detail, an apparent assumption by the author (p. 40) of a false theorem to the effect that a sequence of analytic functions which converges at every point of a region necessarily converges uniformly in every compact subset of the region.

W. Strodt (New York)

152:

Kurepa, Svetozar. A cosine functional equation in Hilbert space. Canad. J. Math. **12** (1960), 45–50.

The paper is concerned with the functional equation $F(x+y)+F(x-y)=2F(x)F(y)$, where x and y are real variables and F is a mapping of the real line R into the Banach space $L(H)$ of all continuous linear operators on the Hilbert space H. The following two theorems are proved. (1) If F satisfies the functional equation on $R\times R$ and is weakly measurable on some interval, then F is weakly continuous, provided (a) that H is separable, and (b) that $F(x)f=0$ a.e. implies $f=0$. (2) If F satisfies the functional equation, if F is weakly continuous, and if $F(x)$ is normal for each x, with the property that $F(x)f=0$ a.e. implies $f=0$, then F has the form $F(x)=\cos(xN)$ (defined by the usual infinite series), where N is a bounded normal operator.

A. E. Taylor (Los Angeles, Calif.)

SEQUENCES, SERIES, SUMMABILITY

153:

Petersen, G. M. Summability and bounded sequences. Proc. Cambridge Philos. Soc. **55** (1959), 257–261.

Ist A_k eine Folge von regulären Matrixverfahren, die so beschaffen sind, dass jede beschränkte und A_k-summierbare Folge auch A_{k+1}-summierbar ist, und summiert ein reguläres Matrixverfahren A alle A_k-summierbaren beschränkten Folgen ($k=1, 2, \cdots$), so gibt es entweder eine beschränkte A-summierbare Folge, die von keinem A_k summiert wird, oder es ist A b-äquivalent mit einem A_k. Sind reguläre Matrixverfahren A_k ($k=1, 2, \cdots$) so beschaffen, dass jede beschränkte und A_{k+1}-summierbare Folge auch A_k-summierbar ist, und summiert jedes A_k wenigstens eine beschränkte divergente Folge, so gibt es eine beschränkte divergente Folge, die von allen A_k summiert wird. Vgl. hierzu A. Broudno [Mat. Sb. (N.S.) **16** (58) (1945), 191–247; MR **7**, 12].

A. Peyerimhoff (Marburg)

154:

Ramanujan, M. S. The problem of "total translativity" for Hausdorff methods. J. Indian Math. Soc. (N.S.) **22** (1958), 45–51.

Ein reguläres Summierungsverfahren A heisst total translativ, wenn aus der A-Summierbarkeit von $\{s_n\}$ gegen l ($l=\pm\infty$ eingeschlossen) stets folgt, dass $\{0, s_0, s_1, \cdots\}$ ebenfalls zum Wert l summierbar ist. Der Verf. zeigt, dass in einer umfangreichen Klasse von Hausdorffverfahren ("satisfied by all the common methods") die Cesàroverfahren die einzigen regulären und total translativen Verfahren sind.

A. Peyerimhoff (Marburg)

155:

Parameswaran, M. R. Some remarks on Borel summability. Quart. J. Math. Oxford Ser. (2) **10** (1959), 224–229.

Let M denote a conservative Hausdorff method (H, μ_n) or a quasi-conservative Hausdorff method (H^*, μ_n), and let B denote the Borel exponential method. It is known that in the space of bounded sequences the inclusion relation $M \supset B$ holds if and only if $\mu_n \to 0$. The author shows that if $\mu_n \not\to 0$ and M sums a bounded sequence s which is summable (B), then s converges. He also proves some Tauberian theorems concerning the "Hölder combinations" $H^\alpha - cH^{\alpha+1}$ (α real, c complex).

G. Piranian (Ann Arbor, Mich.)

APPROXIMATIONS AND EXPANSIONS

See also B510.

156:

Cot, Donatien. ★Eléments des calculs d'interpolation. Publ. Sci. Tech. Ministère de l'Air, Bull. Serv. Tech. no. 123, Paris, 1959. xi+140 pp. 4.435 francs.

Exposé didactique avec exemples des méthodes classiques d'interpolation pour les fonctions d'une ou plusieurs variables. Indications sur les formules de dérivation et de quadrature approchée.

J. Kuntzmann (Grenoble)

157:

Matsuyama, Noboru. Notes on gap theorems. Sci. Rep. Kanazawa Univ. **6** (1959), 61–64.

The author extends the results of a former paper [same Rep. **6** (1958), 1–7; MR **21** #1480] by replacing the gap series considered there by the general one (1) $\sum_{k=1}^{\infty} a_k f(n_k x)$, where $f(x) \in L^2$, and the $n_1 < n_2 < \cdots$ are natural numbers. Under a condition concerning the magnitude of $|a_k|$ and a condition concerning the magnitude of $|c_k|$, the c_k's being the Fourier coefficients of $f(x)$, he proves the almost everywhere convergence of (1). Essentially the same method is used.

J. F. Koksma (Amsterdam)

158:

Butzer, Paul-L. Sur le rôle de la transformation de Fourier dans quelques problèmes d'approximation. C. R. Acad. Sci. Paris **249** (1959), 2467–2469.

Let $f \in L(-\infty, \infty)$ and let

$$T\rho(x) = (2\pi)^{-1/2}\rho \int_{-\infty}^{\infty} f(x+u)k(\rho u)du,$$

where $k \in L$, k is even and positive, $(2\pi)^{-1/2}\int k(u)du = 1$, and either $k(u)$ decreases on $(0, \infty)$ or $k(u)$ is dominated by an element of L. Then $T_\rho(x) \to f(x)$ in the L-metric as $\rho \to \infty$. The author states the following theorem. Suppose that for some c, γ, λ we have $\lim_{\rho\to\infty} \rho^\gamma(1-\hat{k}(\nu/\rho)) = c|\nu|^\lambda$ for every real ν, where $\hat{k}$ is the Fourier transform of k. If $\|T_\rho(x) - f(x)\|_L = o(\rho^{-\gamma})$, then $f(x) = 0$ almost everywhere; if the same norm is $O(\rho^{-\gamma})$, then there is a function g of bounded variation such that $c|\nu|^\lambda \hat{f}(\nu) = \hat{g}(\nu)$ (Fourier-Stieltjes transform of g). Applications to special singular integrals.

R. P. Boas, Jr. (Evanston, Ill.)

159:

Friedman, B. Stationary phase with neighboring critical points. J. Soc. Indust. Appl. Math. **7** (1959), 280–289.

Simplified exposition of the results of an earlier paper [C. Chester, B. Friedman, and F. Ursell, Proc. Cambridge Philos. Soc. **53** (1957), 599–611; MR **19**, 853].

P. Henrici (Los Angeles, Calif.)

FOURIER ANALYSIS

See also 91, 158.

160:

Gosselin, Richard P. On Diophantine approximation and trigonometric polynomials. Pacific J. Math. **9** (1959), 1071–1081.

Le résultat principal concerne la divergence des polynômes trigonométriques d'interpolation. On désigne par $I_{n,u}(x; f)$ le polynôme trigonométrique d'ordre n prenant les mêmes valeurs que f aux points $u + 2k\pi(2n+1)^{-1}$, et par ψ une fonction croissante définie sur $(0, \infty)$. Théorème: Il existe une fonction f telle que $\psi(|f|) \in L^1(0, 2\pi)$ et telle que la suite $I_{n,u}(x; f)$ diverge p.p. dans le carré $0 \leqq x \leqq 2\pi$, $0 \leqq u \leqq 2\pi$. Pour $\psi(x) = x^p$ $(1 \leqq p < 2)$, on retrouve un résultat de Marcinkiewicz et Zygmund [Fund. Math. **28** (1936), 131–166]. Un théorème auxiliaire concerne la mesure de l'ensemble des points $x \in [0, 1]$ qui satisfont la condition suivante (où $\gamma > 0$ et m entier sont fixés): il existe une fraction irréductible p/q, telle que $\gamma m < q \leqq m$ et $|x - p/q| \leqq 1/\gamma m^2$ [resp. $2/\gamma^2 m^2$].

J.-P. Kahane (Montpellier)

161:

Šladkowska, Janina. Sur l'ensemble des points de divergence des séries de Fourier des fonctions continues. C. R. Acad. Sci. Paris **250** (1960), 258–259.

The paper contains the following two theorems. (1) If E is a subset of the interval $0 < x \leqq 2\pi$, E of type $G_{\delta\sigma}$, $E \subset M$, where M is of type F_σ and of logarithmic measure zero, then there exists a continuous function $f(x)$ whose Fourier series has uniformly bounded partial sums, diverges in E and converges elsewhere. (2) If $E \subset (0, 2\pi]$, E of type G_δ, $\bar{E} \cdot \mathscr{G}$ of logarithmic measure zero, where $\mathscr{G}$ is an open set, then there exists a continuous function $f(x)$ whose Fourier series diverges with infinite oscillation for each x in E and converges elsewhere. In the proof it is briefly sketched how such functions $f(x)$ can be constructed. The theorems represent extensions of results previously obtained by Erdös, Herzog and Piranian [Math. Scand. **2** (1954), 262–266; MR **16**, 691].

F. Herzog (E. Lansing, Mich.)

162:

Varshney, O. P. On the absolute harmonic summability of a series related to a Fourier series. Proc. Amer. Math. Soc. **10** (1959), 784–789.

Let $\sum a_n$ be an infinite series and s_n be its nth partial sum. If the sequence $\{(\log n)^{-1}\sum_{k=0}^{n}(k+1)^{-1}s_{n-k}\}$ is of bounded variation, then the series $\sum a_n$ is said to be absolutely summable by harmonic means. Further let f be an integrable function and its Fourier series be

$$\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

Then the author proves that if $\varphi(t)=\frac{1}{2}[f(x+t)+f(x-t)]$ is of bounded variation in the interval $(0,\pi)$, then the series

$$\sum_{n=1}^{n}(a_n\cos nt+b_n\sin nt)/\log(n+1)$$

is absolutely summable by harmonic means.

S. Izumi (Sapporo)

163:

Sunyer i Balaguer, Ferran. Sur des cas où l'inégalité fondamentale de M. S. Mandelbrojt peut être précisée. C. R. Acad. Sci. Paris **249** (1959), 2472–2474.

The author shows that for a sequence $\{\lambda_n\}$ such that $\sum 1/\lambda_n$ converges both rapidly and regularly, he can sharpen the conclusion both of Mandelbrojt's fundamental inequality [*Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952; MR **14**, 542] and of a supplementary result of his own [Collect. Math. **5** (1952), 240–267; MR **15**, 694].

R. P. Boas, Jr. (Evanston, Ill.)

164:

Koosis, Paul. Sur un théorème remarquable de M. Malliavin. C. R. Acad. Sci. Paris **249** (1959), 352–354.

The author refines the estimation technique of Malliavin [same C. R. **248** (1959), 1756–1759, 2155–2157; MR **21** #5854a, #5854b] so as to obtain a counter-example to spectral synthesis in the integers involving an absolutely convergent Fourier series with Hadamard gaps, e.g., $a+\sum_1^\infty n^{-2}\sin\lambda_n t$ where $\lambda_n^{-1}\lambda_{n+1}$ is bounded. In Malliavin's construction this quantity increases geometrically.

C. S. Herz (Ithaca, N.Y.)

165:

Katznelson, Y. Sur le calcul symbolique dans quelques algèbres de Banach. Ann. Sci. École Norm. Sup. (3) **76** (1959), 83–123.

Let B be a commutative semi-simple strictly real Banach algebra with 1, considered as a pointwise algebra of real functions on its maximal ideal space $\mathcal{M}$. Let F be a real function with domain some real interval K. Then F "operates on B" if the composite function $F(g)$ belongs to B for every $g\in B$ whose spectrum lies in K.

In Part I the author considers the algebra $\mathcal{F}_0$ of all F [resp. the algebra $\mathcal{F}$ of all 2π-periodic F] that operate on B, taking K as the whole real line. Theorem: If $\mathcal{F}_0$ contains a discontinuous function, then B is finite-dimensional. Theorem: Let B be regular in the sense of Šilov, and let $F\in\mathcal{F}_0$. Then for each $m\in\mathcal{M}$, with finitely many exceptions, there is some neighborhood V of m, and some $\varepsilon>0$, such that whenever $g\in B$ has support in V and $\|g\|<\varepsilon$, then $\|F(g)\|<1/\varepsilon$.

The next four theorems assume there exists some norm making $\mathcal{F}$ a Banach algebra. Theorem: $\mathcal{F}$ contains all p-times-differentiable functions, for some p. In particular, $\mathcal{F}$ is regular in the sense of Šilov. Theorem: If $\mathcal{F}$ has no closed primary ideals, then B is the full algebra $C(\mathcal{M})$ of all continuous functions on $\mathcal{M}$. Theorem: If $\mathcal{F}$ contains some $F=t^{1/2}$ for small positive t, then $B=C(\mathcal{M})$. {Since the publication of the present paper, the author has discovered a proof of this square-root theorem that avoids the assumption of a Banach algebra structure for $\mathcal{F}$.} Theorem: Let w_n be the norm in $\mathcal{F}$ of the function e^{inx}, and let $A\{w_n\}$ be the algebra of all F whose Fourier coefficients satisfy $\sum|c_n|w_n<\infty$. Then $A\{w_n\}$ is strictly smaller than $\mathcal{F}$. (Sketch of proof. It is easy to find F_0 such that $F\to F(F_0)$ is an endomorphism of $\mathcal{F}$. For instance any $F_0\in\mathcal{F}$ will do. But, supposing F_0 twice differentiable and w_{2n}/w_n bounded, then $F\to F(F_0)$ cannot be an endomorphism of $A\{w_n\}$ unless F_0 is linear. This generalizes results of Beurling and Helson and Leĭbenzon.)

In the latter half of the paper the author applies his general theory to the case where B is the algebra A of real functions with absolutely convergent Fourier series, or the restriction A_K of A to some closed subset K of the real line. Theorem (converse of Wiener-Lévy): F operates on A only if F is real-analytic. More generally, the same holds for A_K if K has positive measure, or even if K contains arbitrarily long arithmetic progressions. Theorem (also proved by Kahane and Salem): Let $g\in A$ be periodic, of bounded variation, and Lipschitzian of some positive order. Then g can be approximated in norm by $g_n\in A$ vanishing on neighborhoods of the zeroes of g. Theorem: For all $g\in A_K$ and some $\alpha>1$, suppose $\|e^{ing}\|=O(|n|^\alpha)$. Then K is a Helson set, i.e., $A_K=C(K)$.

H. Mirkil (Hanover, N.H.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

166:

Griffith, James L. On some aspects of integral transforms. J. Proc. Roy. Soc. New South Wales **93** (1959), 1–9.

This is an expository article in which the author surveys the present state of the theory of integral transforms, and ventures some predictions as to future developments. Particular attention is paid to transforms arising in applied mathematics, and to generalizations of such transforms.

P. G. Rooney (Toronto)

167:

Chover, J. A theorem on integral transforms with an application to prediction theory. J. Math. Mech. **8** (1959), 939–945.

The main theorem proved is as follows: Theorem I. Let (i) r be a non-negative, continuous function on the closed interval $[a-b,b-a]$, $a<b$; (ii) r be non-decreasing on $[a-b,0]$ and non-increasing on $[0,b-a]$; (iii) the left- and right-derivatives r^-, r^+ of r exist and be finite and non-decreasing on both the open intervals $(a-b,0)$, $(0,b-a)$; (iv) $\int_a^b r(t-s)m(ds)\not\equiv 0$ for $t\in[a,b]$, for any non-zero Radon measure m on $[a,b]$. Then the necessary and sufficient condition that a continuous, complex-valued function f on $[a,b]$ should have a representation

$$(1)\qquad f(t)=\int_a^b r(t-s)dg(s)\quad(a\le t\le b),$$

with some function g of bounded variation on $[a,b]$, is that the left-derivative f^- of f be of bounded variation on (a,b).

Its surprising feature is that while (1) involves both f and r, the necessary and sufficient condition given for (1) involves f alone (assuming, of course, that r satisfies (i)–(iv)). Taking, in particular, $f(t)=r(t-c)$ where $a<b<c$,

it follows from (iii) that f^- is of bounded variation on (a, b), and hence by theorem I that

$$(3) \qquad r(t-c) = \int_a^b r(t-s)dg(s) \quad (a \leq t \leq b),$$

g depending on a, b, c.

This is significant in case r is the covariance function of a weakly stationary, mean-continuous, stochastic process $x(t)$, $-\infty < t < \infty$. For then (3) is a (trivial) necessary and sufficient condition for the linear prediction $\hat{x}(c)$ of $x(c)$ obtainable from the $x(t)$ for $a \leq t \leq b$, to be "autoregressively" expressible as a stochastic integral

$$(2) \qquad \hat{x}(c) = \int_a^b x(t)dg(t).$$

Theorem I shows that such an expression will exist when r satisfies (i)–(iv). It thus sheds light on the important problem of finding nice necessary and sufficient conditions on r or on the spectral distribution of the process which will ensure (2). But it does not solve this problem, for the conditions (i)–(iv) are not necessary; they are not even sufficient in the important case of prediction from the "full past" $(-\infty, 0]$, i.e., when $a = -\infty$.

The proof of theorem I rests on the fact that (1) entails that $m_{f^-}(E) = (v_0 * m_g)(E \cap (a, b)) = \{(Q - \lambda I)m_g\}(E)$, where m_{f^-}, m_g are measures induced on $[a, b]$ by f^-, g; v_0 is the measure induced on $[a-b, b-a]$ by r^-; $*$ refers to convolution of measures; E is a Borel subset of $[a, b]$; $\lambda = r^-(0-) - r^-(0+)$; Q is a linear operator on the space of Radon measures on $[a, b]$ which assign zero mass to a and b; and I the identity operator. By (ii), (iii) $\lambda > 0$, i.e., r has a cusp at 0. This is crucial to the argument the author uses to show that $Q - \lambda I$ is invertible. This enables him to recover m_g and thence g from the given f by the inversion $m_g = (Q - \lambda I)^{-1} m_{f^-}$.

Another theorem, II, concerns the estimation of g in (1), once its existence is guaranteed by theorem I.

P. Masani (Bloomington, Ind.)

168:

Uzelac, Zora. Inversion of the Laplace transform of a class of algebraic functions. Univ. Beograd. Godišnjak Filozof. Fak. Novi Sad **2** (1957), 373–380. (Serbo-Croatian. Russian summary)

Inversion formulae are given for the equations

$$\int_0^\infty e^{-st} f(t)dt = s^{-\mu}\left[\frac{s^{p/q}}{as^{p/q}+1}\right]^k, \quad \mu > 0,\ k > 0,\ p/q > 0;$$

$$\int_0^\infty e^{-st} f(t)dt = s^{-\mu}\left[\frac{s^2}{(1+as^\nu)^2 + b^2 s^2}\right], \quad \mu > 0,\ \nu > 0.$$

A. Devinatz (Princeton, N.J.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

169:

Guo, Da-d'yun'. The stability of solutions of linear integral equations. Sci. Sinica **8** (1959), 331–356. (Russian)

L'A. studia il problema di stabilità delle soluzioni dell'equazione integrale lineare di Fredholm di seconda specia

$$(*) \qquad x(s) - \lambda \int_E K(s, t)x(t)dt = y(s),$$

nel senso della dipendenza continua di essi soluzioni dal nucleo e dal termine libero. I risultati conseguiti si referiscono successivamente allo studio dell'equazione (*) per $E \equiv [0, 1]$ nello spazio C delle funzioni continue, poscia nello spazio $C^p(0, 1)$ delle funzioni che ne possegono derivate d'ordine p continue, quindi nello spazio hilbertiano separabile H ed, infine, negli spazi di Banach $L_p(p > 1)$ e B, primo costituito da tutte le funzioni misurabili su E per i quali $\int_E |x(s)|^p ds < +\infty$ e l'atro ne è lo spazio generale.

D. Mangeron (Iaşi)

170:

Woodward, David A. On a special integral equation. Proc. Amer. Math. Soc. **10** (1959), 853–854.

The question posed by R. H. Cameron [J. Analyse Math. **5** (1956/57), 136–182; MR **19**, 428] whether the integral equation

$$y(t) = x(t) + \int_0^t [x(s)]^2 ds$$

has a solution x in C, the class of continuous functions on $[0, 1]$ vanishing at $t = 0$, for almost all y in C, is answered in the negative by the set of y in C subject to the condition $|y(t) + 4t| < 1/10$, $0 \leq t \leq 1$.

T. H. Hildebrandt (Ann Arbor, Mich.)

FUNCTIONAL ANALYSIS

171:

Schaefer, Helmut. Zur komplexen Erweiterung linearer Räume. Arch. Math. **10** (1959), 363–365.

If M is a real vector space, its complex extension is the product $M \times M$ with scalar multiplication defined by $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x)$. This space, with multiplication restricted to real scalars, is isomorphic to $M + iM$. Theorem 1: If L is a complex vector space, it is the complex extension of one of its real subspaces N. (Any real subspace which is maximal among those for which $N \cap iN = \{0\}$ will suffice.) Theorem 2 gives necessary and sufficient conditions that certain complex topological vector spaces be topologically isomorphic (in their weak topology over the reals) with the topological direct sum $N \oplus iN$.

R. R. Phelps (Berkeley, Calif.)

172a:

Nakano, Hidegorô; Sasaki, Masahumi. Convergence concepts in semi-ordered linear spaces. I. Proc. Japan Acad. **35** (1959), 25–30.

172b:

Nakano, Hidegorô. Convergence concepts in semi-ordered linear spaces. II. Proc. Japan Acad. **35** (1959), 83–88.

In these two papers, the authors define certain general types of weak convergence of sequences of elements in a

semi-ordered linear space and develop relations between certain ones of them. The terminology is that used in the book of Nakano [*Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR **12**, 420]. Let R be a continuous semi-ordered linear space. If $\{a_\nu\}$, $\nu=0, 1, 2, \cdots$, is a sequence of elements of R, let

$$a_0 = \lim_{\nu\to\infty} a_\nu = \bigcap_{\nu=1}^{\infty} \bigcup_{\mu\geq\nu} a_\mu = \bigcup_{\nu=1}^{\infty} \bigcap_{\mu\geq\nu} a_\mu.$$

A mapping $\alpha:\Sigma\to\Sigma$, where Σ is the family of all sequences of elements of R, is an 'operator' if $a_0=\lim_{\nu\to\infty} a_\nu$, implies $a_0^\alpha=\lim_{\nu\to\infty} a_\nu^\alpha$. A set $\mathfrak{A}$ of operators is a 'process' if $a_0=\lim_{\nu\to\infty}$, $b_0=\lim_{\nu\to\infty} b_\nu$, $a_0\neq b_0$, implies that there exists $\alpha\in\mathfrak{A}$ for which $a_0^\alpha\neq b_0^\alpha$. A set A of processes is a 'modificator' if for $\mathfrak{A}_1, \mathfrak{A}_2\in A$ there exists $\mathfrak{A}\in A$ with $\mathfrak{A}\subset\mathfrak{A}_1\cap\mathfrak{A}_2$. The authors define products AB and direct products $A\circ B$ for modificators and show that they are again modificators. For a modificator A, $a_0=A\text{-}\lim_{\nu\to\infty} a_\nu$ if for some $\mathfrak{A}\in A$, $a_0^\alpha=\lim_{\nu\to\infty} a_\nu^\alpha$ for all $\alpha\in\mathfrak{A}$. In the first paper the authors define four canonical modificators, including among others the modificators O consisting of the identity operator and S consisting of mappings of sequences onto their subsequences, and show that if R is super universally continuous [loc. cit.] then any combination of these four canonical types by products or direct products (standard modificators) is equivalent to a product of not more than two. If in addition R is complete [loc. cit.], then any such combination is equivalent to O or S.

In the second paper the author proves theorems of a similar nature for standard modificators and for a more restricted class (simple modificators) if the space satisfies a somewhat weaker condition of local super universal continuity. Examples are given to show how the notions of A-convergence apply to partially ordered function spaces. Thus, if S is a set, m a countably additive measure over a σ-field of subsets of S, it is shown that for certain semi-ordered spaces of real valued measureable functions on S, almost everywhere convergence and convergence in measure are equivalent to A-convergence, the modificator A being chosen as an appropriate simple modificator in each case. *R. E. Fullerton* (College Park, Md.)

173:

Günzler, Hans. Über ein Analogon zum Satz von Hahn und Banach. Arch. Math. **10** (1959), 366–372.

Suppose N is a (real or complex) normed linear space and L a subspace (not necessarily closed) of N^*. Consider the property (E): For each $\varepsilon>0$ and each $x\in N$ there exists $x_\varepsilon\in N$ such that $g(x_\varepsilon)=g(x)$ for all $g\in L$ and $\|x_\varepsilon\|\leq\|x\|_L+\varepsilon$, where $\|x\|_L=\sup\{|g(x)|: g\in L, \|g\|\leq 1\}$. Ivan Singer [C. R. Acad. Sci. Paris **247** (1958), 408–411; MR **20** #5423] has proved that (E) holds whenever L is w^*-closed. The author extends this result, and obtains a converse to the extended version, in the following (Theorem 2): Property (E) holds if and only if the unit sphere U of L is w^*-dense in the unit sphere of the w^*-closure of L. An example shows that this has wider scope than Singer's theorem. A theorem of R. C. James [Ann. of Math. (2) **66** (1957), 159–169; MR **19**, 755] is applied to show (Theorem 4) that for a separable Banach space B, if (E) holds with $\varepsilon=0$ for all w^*-closed subspaces $L\subset B^*$, then B is reflexive. {The proof of "(a') implies (b')" of Lemma 1 can be shortened considerably by observing that if f is not in the w^*-closure of U, then, by the separation theorem, there exists $x\in N$ such that $\|x\|_L<|f(x)|$. Theorem 4 can be extended to non-separable B by showing that for any closed separable subspace M of B, the property (E) holds with $\varepsilon=0$ and L a one-dimensional subspace of M^*; the author's method will then show that M is reflexive, and hence B must be reflexive.} *R. R. Phelps* (Berkeley, Calif.)

174:

Leichtweiss, Kurt. Selbstadjungierte Banach-Räume. Math. Z. **71** (1959), 335–360.

If $\mathfrak{L}$ is a Banach space and $\mathfrak{L}^*$ is its adjoint, there is an obvious canonical isometry of $\mathfrak{L}$ into $\mathfrak{L}^{**}$. The less immediate question of studying the structure of $\mathfrak{L}$ by postulating isometric linear maps of $\mathfrak{L}^*$ onto $\mathfrak{L}$ has received attention. In particular, if A is a linear homeomorphism of norm one for which $f(Af)=\|f\|\,\|Af\|$ ($f\in\mathfrak{L}^*$, $Af\in\mathfrak{L}$, and $f(Af)$ representing the value of the linear functional f on the vector Af), then $\|f\|=\|Af\|$, that is, A is an isometry, and $\mathfrak{L}$ is either a euclidean or a Hilbert space [E. R. Lorch, Ann. of Math. **46** (1945), 468–473; MR **7**, 125]. It is known from the cited work that the existence of a mere isometry between $\mathfrak{L}^*$ and $\mathfrak{L}$ is not sufficient to guarantee the euclidean character of the space. In the present work, the author considers problems to which one is led by the above considerations; thus he studies the structure of spaces $\mathfrak{L}$ which are (in his terminology) self-adjoint, that is, for which there exists at least one isomorphism A from $\mathfrak{L}^*$ onto $\mathfrak{L}$.

For the most part, the spaces (all real!) considered are more general than Banach spaces in that the identity $\|\alpha x\|=|\alpha|\,\|x\|$ is assumed for $\alpha\geq 0$ only. If the dimensionality of $\mathfrak{L}$ is finite, $\mathfrak{L}$ is called a Minkowski space. In general, $\mathfrak{L}$ is called skew-normed. A linear functional f on $\mathfrak{L}$ is bounded if $f(x)\leq\mu$ for all x with $\|x\|=1$. The set $\mathfrak{L}^*$ of all bounded linear functionals is then also a skew-normed space providing one defines $\|f\|=\sup f(x)$, $\|x\|=1$. The standard development for $\mathfrak{L}^*$, $\mathfrak{L}^{**}$ follows. If $\mathfrak{L}$ is self-adjoint with isometry A, it is called symmetric providing that $A^{-1}x(y)=A^{-1}y(x)$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are skew-normed spaces, then $\mathfrak{L}_1\times\mathfrak{L}_2$ is defined to consist of all pairs $x=(x_1, x_2)$ with $\|x\|=(\|x_1\|^2+\|x_2\|^2)^{1/2}$; this product is also skew-normed. It is shown that if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are self-adjoint with the isomorphisms A_1 and A_2 resp., then $\mathfrak{L}_1\times\mathfrak{L}_2$ is self-adjoint with the isomorphism $A_1\times A_2$ (defined in an obvious way). A self-adjoint space with isometry A is symmetric (with respect to A) if and only if $\mathfrak{L}$ is reflexive and $A^*=A$.

From now on, all spaces are finite dimensional (Minkowski) spaces $\mathfrak{M}$. Every isomorphism A thus has matrix representations. It is shown that if $\mathfrak{M}$ is self-adjoint with isometry A, then the matrix of A is congruent to a normal matrix. There follows a detailed analysis of the structure of the possible isomorphisms of $\mathfrak{M}$. It is shown that a self-adjoint $\mathfrak{M}$ of dimension n possesses an isomorphism $A^{(0)}$ whose matrix is congruent to one of the following type:

$$(1+\cdots+1+d_1+\cdots+d_m+(-1)+\cdots+(-1)),$$

where the symbols represent orthogonal matrices along the principal diagonal, the d_j being of dimension two, all others of dimension one. Thus d_j represents a rotation of a two-space by the angle ϕ_j with $\phi_j=l_j\pi/2^{\lambda_j}$ where λ_j is a positive integer, l_j an odd integer. All these matrices can arise from specific self-adjoint Minkowski spaces which the author constructs.

Finally, the author characterises the Minkowski spaces $\mathfrak{M}$ which are euclidean: $\mathfrak{M}$ is euclidean if and only if it possesses an isomorphism whose matrix is symmetric and positive definite. A self-adjoint $\mathfrak{M}$ with completely homogeneous norm and of dimension greater than 2 is euclidean if and only if (essentially) it admits as automorphisms all euclidean motions which leave a line d through the origin pointwise fixed. *E. R. Lorch* (New York)

175:

Sikorski, R. Determinant systems. Studia Math. **18** (1959), 161–186.

This paper was suggested by recent work on the generalization of Fredholm's theory of integral equations.

The author considers two vector spaces Ξ and X in duality over a commutative field F ("conjugate linear spaces" in his terminology). The fundamental bilinear functional setting up the duality is denoted multiplicatively $((\xi, x)\to\xi x)$, and a notation suggested by matrix theory $((\xi, x)\to\xi Ax)$ is used for other bilinear functionals A on $\Xi\times X$. Attention is concentrated on those bilinear functionals A which are associated (in a natural way) both with endomorphisms $(x\to Ax)$ of X and also with endomorphisms $(\xi\to\xi A)$ of Ξ. Multiplication can be defined in an obvious way for these functionals, and they form the algebra $\mathfrak{A}$. In what follows, the author usually further restricts attention to a suitable subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. This subalgebra is assumed to include the identity I, all finite-dimensional functionals, and also the inverse of any element of $\mathfrak{B}$ which has an inverse in $\mathfrak{A}$.

By a "determinant system" in $\mathfrak{B}$ for an element A of $\mathfrak{B}$ the author means a sequence $D_0, D_1, D_2, \cdots$ of multilinear functionals satisfying certain requirements. Here D_n is a $2n$-linear functional on the Cartesian product $\Xi^n\times X^n$. This is the approach of Leżański [Studia Math. **13** (1953), 244–276; MR **15**, 535], and is closely related to that of Grothendieck [Bull. Soc. Math. France **84** (1956), 319–384; MR **19**, 558] and Ruston [Proc. London Math. Soc. (3) **1** (1951), 327–384; MR **13**, 468] (see, for instance, op. cit. § 3.1). Each element D_n of the sequence is skew symmetric in each of the sets of n variables, and when regarded as a function of one variable of each set (the remaining variables being kept constant) belongs to $\mathfrak{B}$. At least one of the D_n is not identically zero. The remaining requirements consist of algebraic relations (in the reviewer's notation [loc. cit.] they are $A\{1\}D_{n+1}=I\wedge D_n$ and $D_{n+1}\{1\}A=I\vee D_n$). The author proves that A has a determinant system if and only if the two associated endomorphisms have ranges of the same finite codimension, and that the determinant system is then unique apart from a constant non-zero numerical factor. He finds a quasi-inverse B for A (i.e., a bilinear functional such that $ABA=A$ and $BAB=B$) in terms of the determinant system.

The paper concludes with a discussion of determinant systems in product spaces.

{Notes: There is a rather large number of minor misprints in this paper. Some of the author's terminology strikes the reviewer as curious. For instance, a one-dimensional bilinear functional is defined (p. 163) in such a way that the zero functional is not excluded, and a projection is said to be r-dimensional (p. 165) if the range of the associated endomorphism is of codimension r.} *A. F. Ruston* (Sheffield)

176:

Moppert, C. F. On the Gram determinant. Quart. J. Math. Oxford Ser. (2) **10** (1959), 161–164.

Let P be a projection operator in a Hilbert space H. Let $a, a_i\in H$ and $b=Pa$, $b_i=Pa_i$ $(1\leq i\leq n)$. Let A denote the linear manifold spanned by $a_1, a_2, \cdots, a_n$; and B the linear manifold spanned by $b_1, b_2, \cdots, b_n$. Let $G(a_1, a_2, \cdots, a_n)$ denote the Gram determinant of $a_1, a_2, \cdots, a_n$. The following results are proved. (I) The distance from b to B is at most equal to the distance from a to A. They are equal if and only if $a-a_A=b-Pa_A$, where $a_A\in A$ and $\|a-a_A\|=\min_{x\in A}\|a-x\|$.

$$\text{(II)} \qquad G(a_1, a_2, \cdots, a_n) \geq G(b_1, b_2, \cdots, b_n),$$

with equality if and only if $a_i=b_i$ $(1\leq i\leq n)$.

$$\text{(III)} \qquad \frac{G(b_1, b_2, \cdots, b_n)}{G(b, b_1, b_2, \cdots, b_n)} \geq \frac{G(a_1, a_2, \cdots, a_n)}{G(a, a_1, a_2, \cdots, a_n)}.$$

It is pointed out that two earlier results of W. N. Everitt [Quart. J. Math. Oxford Ser. (2) **8** (1957), 191–196; MR **20** #1905] appear as special cases of (II), (III).

Ky Fan (Notre Dame, Ind.)

177:

Brown, Arlen. On the absolute equivalence of projections. Bul. Inst. Politehn. Iaşi (N.S.) **4** (8) (1958), no. 3–4, 5–6. (Russian and Romanian summaries)

Two projections E and F on Hilbert space $\mathfrak{H}$ are said to be absolutely equivalent if the ring generated by E and F contains an element W with $W^*W=E$, $WW^*=F$. It follows from results of J. Dixmier [Revue Sci. **86** (1948), 387–399; MR **10**, 546] that E and F are absolutely equivalent if and only if

$$\text{(1)} \qquad E\wedge(1-F) = 0 \quad\text{and}\quad F\wedge(1-E) = 0.$$

This is the case in particular if

$$\text{(2)} \qquad \|E-F\| < 1,$$

and then the operator

$$V = F[1+E(F-E)E]^{-1/2}E,$$

considered by the reviewer [Comm. Math. Helv. **19** (1947), 347–366; MR **8**, 589], may be chosen for W. The author shows that even in the general case (1) the operator V may be formed as a densely defined and bounded operator, and the extension $\bar{V}$ of V to the whole space $\mathfrak{H}$ by closure yields always the operator W in question. Moreover, in case (1), V is defined itself on the whole space $\mathfrak{H}$ only if (2) holds. {Related results may be found also in the paper of C. Davis, Acta Sci. Math. Szeged. **19** (1958), 172–187 [MR **20** #5425].} *B. Sz.-Nagy* (Szeged)

178:

Kwan, Chao-chih. Sur l'espace linéaire semi-ordonné formé des fonctions jouissant de la propriété de Baire. Advancement in Math. **4** (1958), 457–461. (Chinese. French summary)

Résumé de l'auteur: "M. T. B. Yang [Acta Math. Sinica **8** (1958), 95–101; MR **20** #3939] a démontré que les fonctions définies sur l'intervalle [0, 1], jouissant de la propriété de Baire et ne prenant de valeurs infinies que sur un sousensemble de première catégorie, modulo les fonctions égales à zéro sauf sur un sousensemble de première catégorie, forment un espace de Riesz complètement

réticulé continu (au sens de M. Kantorovitch), designé par $\mathfrak{B}/\mathfrak{N}$. M. Yang a indiqué que cet espace est tel qu'on ne sait pas si l'on puisse le munir d'une métrique. Ici d'un résultat de M. Floyd on déduit que cet espace n'admet pas une métrique au sens de M. Kantorovitch, et que de plus il n'y a pas de fonctionnelles (0)-linéaires non nulles définies sur cet espace. En se servant des résultats connus on donne une autre démonstration, moins directe sans doute, pour le théorème mentionné ci-dessus de M. Yang. On indique une erreur, incidentalement, dans le livre de M. G. Birkhoff [*Lattice theory*, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed. Amer. Math. Soc., New York, 1948; MR **10**, 673] en fournissant un contre-exemple. M. Yang nous a communiqué un exemple qui montre aussi que l'espace $\mathfrak{B}/\mathfrak{N}$ n'est pas métrisable lorsqu'il est muni de la convergence (0)."

179:

Brace, John W. The topology of almost uniform convergence. Pacific J. Math. **9** (1959), 643–652.

Let $\mathcal{G}(S, F)$ be a linear space of bounded functions of a set S into a locally convex topological linear space F. The author shows the following. (1) The topology of almost uniform convergence on S (i.e., convergence "on every ultrafilter of S"; cf. the same author, Portugal. Math. **14** (1956), 99–104 [MR **18**, 140]) is a locally convex topology on $\mathcal{G}$ and is Hausdorff if F is so. (2) If S is a compact space and functions in $\mathcal{G}$ are continuous, then for any dense subset S' of S the topology of almost uniform convergence on S' coincides in $\mathcal{G}$ with that of point-wise convergence on S. (3) As a consequence of (2), the notion of almost uniform convergence is useful to express weak topologies on topological linear spaces referring not to all but only to some subsets of functionals.

I. G. Amemiya (Tokyo)

180:

Fréchet, Maurice. L'espace des courbes est-il un espace de Banach? C. R. Acad. Sci. Paris **250** (1960), 248–249.

Let Γ be the set of continuous curves in R_3, and define the distance between two curves $\xi, \eta \in \Gamma$ to be $\inf_h d_h$, where d_h is the maximum of the distances between the points of ξ and η that correspond under the homeomorphism h of ξ onto η. It is observed that if the resulting metric space is homeomorphic to the metric space C of continuous functions on $[0, 1]$ then Γ can be made into a Banach space in a natural way.

C. W. Kohls (Rochester, N.Y.)

181:

Gagliardo, Emilio. Ulteriori proprietà di alcune classi di funzioni in più variabili. Ricerche Mat. **8** (1959), 24–51.

Indicando, con Sobolev, $W_p^r(\Omega)$ lo spazio delle funzioni $u(x_1, \cdots, x_n) \in L^p(\Omega)$ insieme alle derivate (nel senso della teoria delle distribuzioni) dei primi r ordini ($p \geqq 1$, $r \geqq 0$) con la norma

$$\|u\|_{W_p^r(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{s \leqq r} \|D^s u\|_{L^p(\Omega)}$$

$$\left(\|D^s u\|_{L^p(\Omega)} = \sum_{|\alpha|=s} \left\| \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right\|, \ \alpha \equiv (\alpha_1, \cdots, \alpha_n), \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \right),$$

l'A. trova alcune maggiorazioni che generalizzano quelle di Sobolev [cfr. anche Gagliardo, Ricerche Mat. **7** (1958), 102–137; MR **21** #1526], che forniscono delle interessanti relazioni di inclusione fra gli spazi $W_p^r(\Omega)$. Ricorderemo qui la seguente maggiorazione:

$$\|D^s u\|^q_{L^q(\Omega)} \leqq c\{\|u\|^{p_0}_{L^{p_0}(\Omega)} + \|D^r u\|^{p_r}_{L^{p_r}(\Omega)} + 1\}$$

ove $0 < s < r$ e $q < rp_0p_r/(sp_0 + (r-s)p_r)$. Maggiorazioni analoghe sono state comunicate da L. Nirenberg al congresso internazionale di Edinburgo; l'A. confronta i suoi risultati con quelli di Nirenberg.

G. Stampacchia (Genoa)

182:

McWilliams, R. D. On projections of separable subspaces of (*m*) onto (*c*). Proc. Amer. Math. Soc. **10** (1959), 872–876.

The author extends a result of A. Sobczyk [Bull. Amer. Math. Soc. **47** (1941), 938–947; MR **3**, 205] on projections in the space (m) which consists of bounded real sequences, with norm the bound. There is no projection of (m) onto (c) the space of convergent sequences or onto (c_0) the sequences which converge to zero. Sobczyk showed that if B is a separable subset of (m) which contains (c_0) then there exists a projection of norm 2 onto (c_0). The author shows that if B is a separable subset of (m) which contains (c), then there is a projection of B onto (c) of norm not exceeding 3 and that there is a B of this kind for which the minimum norm is 3.

F. J. Murray (New York)

183:

Putnam, C. R. On Toeplitz matrices, absolute continuity, and unitary equivalence. Pacific J. Math. **9** (1959), 837–846.

Let (c_n) $(n = 0, \pm 1, \cdots)$ be a real sequence such that $f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}$ is real, $\sum c_n^2 < \infty$. Let T be the matrix (c_{i-j}), $K = (c_{i+j})$, $F = (f_{ij})$ with

$$f_{ij} = 2\pi^{-1} \int_0^\pi f(\theta) \sin i\theta \sin j\theta d\theta.$$

The essential range of f is the set of numbers λ such that for all $\varepsilon > 0$ $|f(\theta) - \lambda| < \varepsilon$ holds on a set of positive measure; m, M are the essential infimum and supremum of f over $(0, \pi)$. It is proved that if K is bounded, then T is self-adjoint. If K is completely continuous the essential range of f is $[m, M]$. An operator $T = \int \lambda dE(\lambda)$ is said to be absolutely continuous if every set of Lebesgue linear measure zero has zero measure in the measure defined by E. If F is absolutely continuous, so is T. The Hilbert matrix $(|i-j|^{-1})$ is absolutely continuous. If $c_n < \alpha^n$ for some α, $0 < \alpha < 1$, T and F are unitarily equivalent; this is true also under some weaker conditions on the c's.

J. L. B. Cooper (Cardiff)

184:

Kilpi, Yrjö. Über selbstadjungierte Fortsetzungen symmetrischer Transformationen im Hilbertschen Raum. Ann. Acad. Sci. Fenn. Ser. A.I. no. 264 (1959), 23 pp.

Let S be a symmetric operator in a Hilbert space $\mathfrak{H}$ with domain $D(S)$, such that either (i) $(Sf, f) \geqq c(f, f)$, or (ii) $\|Sf\|^2 \geqq c^2 \|f\|^2$, for all $f \in D(S)$, where c is some positive constant. The author gives another proof of the fact that there exists a self-adjoint extension H of S having the property $(Hg, g) \geqq c(g, g)$ $(g \in D(H))$ in case (i), or the

property $\|Hg\|^2 \geq c^2\|g\|^2$ ($g \in D(H)$) in case (ii). Also presented are necessary and sufficient conditions in order that the extension H be unique. These conditions are different from those given earlier by M. G. Kreĭn. The proofs use the von Neumann extension theory.

E. A. Coddington (Los Angeles, Calif.)

185:

Sikorski, R. On Leżański endomorphisms. Studia Math. **18** (1959), 187–189.

In an earlier paper [Studia Math. **16** (1957), 99–112; MR **20** #4192] the author gave an example of a Leżański endomorphism T (an operator to which Leżański's theory can be applied) which was not compact, and so could not belong to the trace class. Its square T^2, however, was compact. The author now proves that the square of any Leżański endomorphism is compact. The proof uses the theory of entire functions of finite order.

A. F. Ruston (Sheffield)

186:

Bonsall, F. F. The iteration of operators mapping a positive cone into itself. J. London Math. Soc. **34** (1959), 364–366.

Let X be a partially ordered real Banach space with positive cone X^+ and let T be a compact linear operator in X which maps X^+ into itself and such that

$$\rho = \lim_{n\to\infty} \|T^n\|^{1/n} > 0.$$

Let $A = T - \rho I$ and let ν be the index of ρ for T. Let P, Q be respectively the complementary continuous projections of X onto the null space N and the range R of A^ν. It is shown that in the uniform operator topology,

$$\lim_{n\to\infty} \|(I+T)^n\|^{-1}\cdot(I+T)^n = \|A^{\nu-1}P\|^{-1}\cdot A^{\nu-1}P.$$

R. E. Fullerton (College Park, Md.)

187:

Silverman, R. J.; Yen, Ti. Characteristic functionals. Proc. Amer. Math. Soc. **10** (1959), 471–477.

Let Γ be a semi-group of positive linear operators on a partially ordered real normed linear space E. Suppose there is a u in the interior K^0 of the positive cone K with $\Gamma(u) \subset K^0$. Kreĭn and Rutman [Amer. Math. Soc. Transl. no. 26 (1950); MR **10**, 256] showed that if Γ is commutative then there are characteristic positive functionals, i.e., $f \in E^*$ with $fA = \lambda_A f$, $\lambda_A > 0$, for all $A \in \Gamma$. Existence of a characteristic positive functional is equivalent to existence of a closed hyperplane H disjoint from K for which $\Gamma(H) \subset H$, and this implies existence of a positive functional f for which $f(ABu) = f(Au)f(Bu)$ for $A, B \in \Gamma$, which in turn implies existence of a multiplicative homomorphism $A \to \lambda_A$ of Γ into the positive real numbers and a positive $f \in E^*$ and $m, M > 0$ such that $m\lambda_A \leq f(Au) \leq M\lambda_A$. The principal lemma of the paper shows that if Γ possesses a right invariant mean then the last condition implies the existence of an invariant positive functional ($fA = f$, all $A \in \Gamma$). This provides a new proof of a theorem of Civin and Yood [Pacific J. Math. **6** (1956), 231–237; MR **18**, 221] concerning the existence of invariant positive functionals on left solvable semi-groups. It is also shown that if Γ is a locally finite group then Γ has an invariant positive functional, but this theorem may fail for locally finite semi-groups.

E. Kallin (Berkeley, Calif.)

188:

Régnier, André. Théorèmes ergodiques. C. R. Acad. Sci. Paris **249** (1959), 2149–2150.

Ergodic theorems for a semigroup T_s of continuous linear operators in a locally convex topological linear space $\mathfrak{X}$. Setting $S_t x = t^{-1}\int_0^t T_s x\,ds$, it is only assumed that $\lim_{t\to\infty} (T_s - T_s S_t)x = 0$ for every $x \in \mathfrak{X}$ and every s. An element x is ergodic when $\lim_{t\to\infty} S_t x$ exists. Typical result: x is ergodic if and only if the intersection of the subspace of ergodic elements with the closure of the set of convex linear combinations $\sum \lambda_s T_s x$, $\sum \lambda_s = 1$, is not empty. Corollary 2: If $\mathfrak{X}$ is a Banach space and $\|T_s\| \leq 1$ for all s, x is ergodic if and only if the subspace of all $x^* \in \mathfrak{X}^*$ such that $S_t{}^* x^* \to 0$ is closed. The results are closely related to those of W. F. Eberlein [Trans. Amer. Math. Soc. **67** (1949), 217–240; MR **12**, 112].

G.-C. Rota (Cambridge, Mass.)

189:

Mullikin, Thomas W. Semi-groups of class (C_0) in L_p determined by parabolic differential equations. Pacific J. Math. **9** (1959), 791–804.

The author considers the differential operator $A = p(x)D^2 + q(x)D + r(x)I$, where $D = d/dx$ and p, q and r are defined on the finite interval $[a, b]$. In addition $p > 0$, q and r may be taken to be complex, and p, q and r are in $L_\infty[a, b]$. The operator A is defined precisely in $L_p[a, b]$ by taking its domain as $D(A) = \{u : u, u'$ absolutely continuous and $u, u', u'' \in L_p\}$. If

$$\pi(u) = M_{i1}u(a) + N_{i1}u(b) + M_{i2}u'(a) + N_{i2}u'(b),$$

then define A_π as the restriction of A to the domain $D(A_\pi) = D(A) \cap \{u : \pi(u) = 0\}$.

The Cauchy problem (in this case for a parabolic partial differential equation) is to determine those A_π which generate semi-groups of class (C_0) in $L_p[a, b]$. A semi-group $\{S_t\}$ is of class (C_0) if $\lim_{t\to 0+} S_t u = u$. The main result is the following: If π is regular, A_π is the infinitesimal generator of a semi-group of class (C_0) in $L_p[a, b]$, $1 \leq p < \infty$. A partial converse of this result is obtained for the special operators $\Omega_\pi = D^2$.

The method of proof consists in obtaining estimates on the L_1 and L_2 norms of the powers of the resolvents of A_π. An application of the Riesz-Thorin theorem together with adjoint semi-group theory gives estimates on these norms for $1 \leq p < \infty$. An application of the Feller-Phillips-Miyadera theorem completes the proof.

A. Devinatz (Princeton, N.J.)

190:

Bishop, Errett. Some theorems concerning function algebras. Bull. Amer. Math. Soc. **65** (1959), 77–78.

Summary of results given in #191 below.

J. Wermer (Providence, R.I.)

191:

Bishop, Errett. A minimal boundary for function algebras. Pacific J. Math. **9** (1959), 629–642.

Let X be a compact Hausdorff space and A an algebra of continuous complex-valued functions on X separating points on X. For each f in A, $\|f\|$ denotes the maximum of $|f|$ on X. Call a subset X' of X a 'boundary' for A if for each f in A there is a point x in X' with $|f(x)| = \|f\|$. Šilov has proved that among all closed boundaries for A there exists a smallest one, now called the 'Šilov boundary'.

The author studies the question of the existence of a smallest, not necessarily closed, boundary for A. He first proves the following striking result. Theorem 1: Assume that X is metrizable and that A is closed under uniform convergence. Then there exists a smallest boundary M for A, and M equals the subset of X consisting of all x such that for some f in A we have $|f(x)|>|f(y)|$ for all $y\neq x$. M is called the 'minimal boundary'. It is clear that the closure of M in X coincides with the Šilov boundary. The author shows by example that M may be a proper subset of the Šilov boundary. He also shows that the hypothesis of metrizability cannot be dropped in theorem 1. Theorem: Under the preceding hypotheses, the minimal boundary M is a countable intersection of open sets. On the way to proving this, the following is shown. Fix a metric q on X and for each x in X set $D_n(x)=\{y: q(x,y)\geqq 1/n\}$, $n=1, 2, \cdots$. Let x be a point such that for each n we can find f in A with $\|f\|\leqq 1$, $|f(x)|>3/4$, $|f(y)|<1/4$ for all y in $D_n(x)$. Then x lies in M. Theorem 3: Maintain the preceding hypotheses and assume 1 lies in A. Let x be a point of X which is not in M. Then there exists a non-negative Borel measure h on $X-\{x\}$ such that $f(x)=\int f dh$, for all f in A. Corollary: Let A be as in theorem 3. Let R be the uniform closure of the space of real parts of functions in A. Let M_0 consist of all points x in X such that there exists g in R with $|g(x)|>|g(y)|$ for all $y\neq x$ in X. Then M_0 coincides with M. The notion of minimal boundary admits an interesting application to questions of uniform approximation in the complex plane. Let E be a compact plane set, and denote by $T(E)$ the class of continuous functions on E which can be uniformly approximated arbitrarily closely by rational functions whose poles lie in the complement of E. Then $T(E)$ is a closed separating algebra of functions on E. Let M be its minimal boundary. Theorem: If E has no interior and if $E=M$, then $T(E)$ contains all functions continuous on E. If $E\neq M$, then in fact $E-M$ has positive 2-dimensional Lebesgue measure. Some further results of a similar nature are also given.

J. Wermer (Providence, R.I.)

192:

Luchins, Edith H. On radicals and continuity of homomorphisms into Banach algebras. Pacific J. Math. **9** (1959), 755–758.

A Banach algebra is said to be absolute if every homomorphism of a Banach algebra into it is continuous, and is said to be strictly semi-simple if its two-sided regular maximal right ideals have zero intersection. It is proved that an absolute Banach algebra contains no non-zero nilpotent elements, and that a strictly semi-simple Banach algebra is absolute. For certain special Banach algebras (including semi-simple annihilator algebras) it is proved that if B contains no non-zero nilpotent elements, then B is strictly semi-simple (and hence absolute).

F. F. Bonsall (Newcastle-upon-Tyne)

193:

Sakai, Shôichirô. On some problems of C^*-algebras. Tôhoku Math. J. (2) **11** (1959), 453–455.

The author proves that (1) if prime (′) denotes a derivation of a C^*-algebra and if x is normal and commutes with x', then $x'=0$; (2) there exists a C^*-algebra with a non-compact pure state space.

C. R. Putnam (Lafayette, Ind.)

194:

Suzuki, Noboru. Certain types of groups of automorphisms of a factor. Tôhoku Math. J. (2) **11** (1959), 314–320; correction **12** (1960), no. 1.

By a modification of his earlier method [Proc. Japan Acad. **34** (1958), 575–579; MR **20** #7233] the author constructs again for each countable group G a faithful representation $g\to\theta_g$ of G onto a group of outer automorphisms of an approximately finite factor $\mathbf{M}$ on a separable Hilbert space. An automorphism of a factor is ergodic if it leaves only the centre elementwise invariant. For each $g\neq e$ in G it is proved that θ_g is ergodic if and only if g is of infinite order.

The special case in which $G=G_1 * G_2$ (free product), with G_1, G_2 countable torsion-free groups, is considered. The author makes use of his theory of the crossed product [same J. (2) **11** (1959), 113–124; MR **21** #4363] to show that the crossed product of $\mathbf{M}$ with the outer automorphic representation of G is in this case of type II_1 and is not approximately finite.

D. A. Edwards (Newcastle-upon-Tyne)

CALCULUS OF VARIATIONS

195:

Janet, Maurice. Sur les systèmes dynamiques à deux degrés de liberté. Bul. Inst. Politehn. Iaşi (N.S.) **4 (8)** (1958), no. 3–4, 103–106. (Russian and Romanian summaries)

This paper is concerned with the extremals of the variational problem

$$\int (\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+\alpha\dot{x}+\beta\dot{y}+u)dt = 0,$$

where α, β, u are functions of x, y. It is shown that by introduction of an appropriate variable φ the variational equations can be brought into the form $\ddot{\varphi}+c\varphi=0$. This is a generalization of similar formulae of H. Poincaré, G. D. Birkhoff and A. Wintner who assumed, in particular, $\alpha_y-\beta_x$ to be a constant.

J. Moser (Cambridge, Mass.)

GEOMETRY

See also 78, 174, B620.

196:

Amir-Moéz, Ali R. Invariance of circle product. Math. Mag. **33** (1959/60), 35–38.

Let C_i be the polynomial $a_i(x^2+y^2)-2b_ix-2c_iy+d_i$. The inner product (C_1, C_2) is defined as $b_1b_2+c_1c_2-(a_2d_1+a_1d_2)/2$. The author asks and answers the question as to what group of transformations of the (x, y)-plane leaves the form of (C_1, C_2) invariant, i.e., leaves (C_1, C_2) invariant for all pairs of circles. Translation, rotation, inversion are included in the group. Every homothety multiplies (C_1, C_2) by the square of the magnification factor. (The entire group is characterized algebraically.)

J. L. Brenner (Palo Alto, Calif.)

197:

Backes, F. L'extension en géométrie cayleyenne et en géométrie anallagmatique de la configuration de Morley-Petersen. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 396–401.

The author considers the following configuration of 20 lines in complex projective 3-space. Let Q be a quadric; a, b, c three lines that are neither generators nor tangents; $\bar{a}$, $\bar{b}$, $\bar{c}$ their polar lines; a', $\bar{a}'$ the 2 transversals to b, $\bar{b}$, c, $\bar{c}$; b', $\bar{b}'$ to c, $\bar{c}$, a, $\bar{a}$; c', $\bar{c}'$ to a, $\bar{a}$, b, $\bar{b}$; a'', $\bar{a}''$ to a, $\bar{a}$, a', $\bar{a}'$; b'', $\bar{b}''$ to b, $\bar{b}$, b', $\bar{b}'$; c'', $\bar{c}''$ to c, $\bar{c}$, c', $\bar{c}'$. Then the 6 lines a'', $\bar{a}''$, b'', $\bar{b}''$, c'', $\bar{c}''$ have a pair of common transversals ω, $\bar{\omega}$ (which, like all the other pairs, are polar lines with respect to Q). Regarding Q as the absolute quadric for a non-Euclidean space, he deduces the Morley-Petersen configuration, which consists of ten pairs of polar lines in elliptic space, or ten real lines in hyperbolic space and so also, by a limiting process, in Euclidean space.

H. S. M. Coxeter (Cedar Falls, Iowa)

198:

Backes, F. Quelques propriétés relatives à une certaine configuration de vingt droites. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 456–461.

The author shows how his configuration of ten pairs of polar lines with respect to a quadric Q (see the preceding review) can be directly related to a Desargues configuration 10_3 in the complex projective plane, namely, in any plane that cuts Q in a conic K. He distinguishes the two systems of generators of Q as "red" and "blue". Any two polar lines, m and $\bar{m}$, are the diagonals of a skew quadrilateral whose sides are two red generators and two blue generators. The two red generators meet K in two points. The pole (with respect to K) of the secant joining these two points is M, the "red image" of m and $\bar{m}$ [cf. H. F. Baker, *Principles of geometry, III*, Cambridge Univ. Press, 1923, p. 68, Ex. 24]. If another pair of polar lines, n and $\bar{n}$, meet both m and $\bar{m}$, their red image N is conjugate to M. Therefore, if a, $\bar{a}$, b, $\bar{b}$ are any two pairs of polar lines, the red image of their two transversals is the pole of AB. Applying this representation to the projective form of the Morley-Petersen configuration, the author obtains two polar triangles ABC, $A'B'C'$, which, by Chasles's Theorem, are perspective triangles. The red images of the six transversals a'', $\bar{a}''$, b'', $\bar{b}''$, c'', $\bar{c}''$ are the poles of the lines AA', BB', CC'. In other words, the Morley-Petersen configuration has for its red image a Desargues configuration (and for its blue image another Desargues configuration in the same plane). Thus the closure of the Morley-Petersen configuration provides a new proof for Chasles's Theorem.

H. S. M. Coxeter (Cedar Falls, Iowa)

199:

Longo, Carmelo. Teorema di Desargues ed omologie speciali in un piano grafico proiettivo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 410–415.

The author discusses certain weak Desarguesian conditions for projective planes. His discussion is of interest because it contributes to the difficult problem of classifying planes.

A projective plane is called $(V, u; a_1; C_1)$-Desarguesian if and only if any triangles $A_1A_2A_3$, $A_1'A_2'A_3'$ perspective from V with $A_1 \in a_1$, $A_1' \in a_1$, and $C_1 = A_2A_3 \cap A_2'A_3'$ must be axial from u when u contains one of the points $C_2 = A_1A_3 \cap A_1'A_3'$, $C_3 = A_1A_2 \cap A_1'A_2'$.

A plane is called (V, u)-transitive if and only if it admits every homology with center V and axis u; (v, u)-transitive if and only if it admits every homology with center on v and axis u.

The author states the following two theorems. (I) Let U, V, W denote distinct collinear points of a projective plane which is $(V, UV; a_1; C_1 = U)$-Desarguesian and $(V, UV; a_2; C_2 = W)$-Desarguesian. Then the plane is (V, UV)-transitive. (II) If a plane is (V, UV)-transitive and $(U, UV; a_1; C_1 = V)$-Desarguesian, it is (UV, UV)-transitive. The proof of theorem I given here covers only the special case $a_1 = a_2$, but theorem II is proved as stated.

The author concludes with applications to coördinates and cartesian systems. Four points O, U, V, E—no three collinear—become the "reference-frame $OUVE$" when those four points are chosen respectively as origin, ideal point on the x-axis, ideal point on the y-axis, and a point on the line which becomes $y = x$. As usual, affine points are identified with ordered pairs (x, y); and affine lines, with $x = c$ and $y = m \cdot x \circ b$. Here the ternary $m \cdot a \circ b$ is the ordinate of the point having abscissa a and lying on the line through $(0, b)$ parallel to $(0, 0) \cup (1, m)$. Put $a + b = 1 \cdot a \circ b$, $ab = a \cdot b \circ 0$, to define addition and multiplication on the (proper) points of one axis. These points form a loop under addition, the loop being associative if and only if the plane satisfies the Reidemeister condition. The ternary satisfies the 'first condition of decomposition', $a \cdot b \circ c = ab + c$, if and only if the plane is $(V, UV; OV; U)$-Desarguesian; it satisfies the 'second condition of decomposition', $m \cdot a \circ (mb) = m(a + b)$, if and only if the plane is $(U, UV; OU; V)$-Desarguesian.

If the ternary satisfies the first condition of decomposition and if one has a group relative to addition, the consequent properties of addition and multiplication characterize a cartesian system. The author observes the following. (1) Necessary and sufficient for the $OUVE$-ternary to give a cartesian system is that the plane be $(V, UV; OV; U)$-Desarguesian and that it satisfy the Reidemeister condition. (2) If the $OUVE$-ternary yields a cartesian system, the plane is (V, UV)-transitive. (3) A necessary and sufficient condition that the $OUVE$-ternary yield a quasifield is that the plane be (V, UV)-transitive and $(U, UV; OU; V)$-Desarguesian [hence, by Theorem II, (UV, UV)-transitive]. (4) If the $OUVE$-ternary satisfies the second condition of decomposition, and if addition is associative, the plane is (U, UV)-transitive.

W. A. Pierce (Syracuse, N.Y.)

200:

Tallini, Giuseppe. Caratterizzazione grafica di certe superficie cubiche di $S_{3,q}$. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 484–489, 644–648.

Define a "cubic" in projective 3-space over a finite field with q elements to be a set of points, not containing any plane or quadric nor consisting of lines through a common point, that contains any quadrisecant line (in particular, any line through two double points, a double point being a point any line through which, if not on the "cubic", meets the "cubic" in at most one extra point). Assuming that q is odd and > 3, and that the "cubic" has at least 3 double points, it turns out that if the cubic

possesses at least the "correct" number of points (depending on q and the configuration of double points), then the inequality is an equality and the "cubic" is a cubic.

M. Rosenlicht (Berkeley, Calif.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

201:

Birch, B. J. On $3N$ points in a plane. Proc. Cambridge Philos. Soc. **55** (1959), 289–293.

The following theorem is proved in this note. Theorem 1: Given $3N$ points in a plane, we can divide them into N triads such that, when we form a triangle with the points of each triad the N triangles will all have a common point.

The proof is given on the basis of three lemmas and two corollaries. The first two lemmas are the fixed-point theorem for n-space and Caratheodory's $(n+1)$-point theorem. Lemma 3 is as follows, where E^n is the unit n-ball: Let a mass distribution in E^n be defined by an integrable density function $\rho(x)$; then we can find a point r inside E^n so that every closed half-space with r on its boundary will contain at least $1/(n+1)$ of the total mass. The first corollary states that Lemma 3 holds if the mass-distribution is not continuous, and the second corollary is as follows: Let Y be a finite set consisting of M points in n-space, and suppose that $M>(n+1)R$. Then there is a point common to all the closed half-spaces which contain at least $(M-R)$ points of Y.

{The author states that he believes Lemma 3 is new. Actually the most important new feature concerns the number and dispositions of closed half-spaces containing an "optimal" portion of the mass stated in the proof. For the Lemma 3 itself, it may be interpreted as a corollary of the finite point-mass problem which is a straightforward generalization to n-space of a theorem stated and proved in Jaglom and Boltjanski [*Konvexe Figuren*, VEB Deutscher Verlag, Berlin, 1956; MR **18**, 146; p. 16], where they also observe that the continuous case is special. To prove Theorem 1, then, the author could have applied his proof directly to the theorem stated by Jaglom and Boltjanski. However, the procedure used and the application of the method to other problems of optimization are of further interest.}

P. C. Hammer (Madison, Wis.)

202:

Beardwood, Jillian; Halton, J. H.; Hammersley, J. M. The shortest path through many points. Proc. Cambridge Philos. Soc. **55** (1959), 299–327.

When $\mathbf{P}$ is a sequence of points in a bounded set E in Euclidean k-space, the authors' main problem is to consider the behaviour of $l(\mathbf{P}^n)$, the length of the shortest path passing through the first n points of the sequence. It is first shown that

$$\text{(A)}\qquad \limsup_{n\to\infty} n^{-(k-1)/k} l(\mathbf{P}^n) \le \alpha_k k^{1/2}[v(E)]^{1/k},$$

where $v(E)$ is the k-dimensional Lebesgue measure of E and α_k is a numerical constant for which bounds are given. The result (A) is an improvement on the result of Verblunsky [Proc. Amer. Math. Soc. **2** (1951), 904–913; MR **13**, 577] in the case $k=2$. If E is open and has a boundary of zero measure, then equality occurs in (A) for certain sequences $\mathbf{P}$ which are said to be maximal: maximal sequences have the property of being uniformly dense in the open subsets of E.

The result (A) is deterministic, giving an exact asymptotic bound for $l(\mathbf{P}^n)$ for all sequences $\mathbf{P}$ in E. The authors are more interested in probabilistic results in which the points $P_1, P_2, \cdots$ are chosen independently according to some probability law from the set E. If a uniform distribution in E is used then, with probability 1,

$$\text{(B)}\qquad \lim_{n\to\infty} n^{-(k-1)/k} l(\mathbf{P}^n) = \beta_k k^{1/2}[v(E)]^{1/k},$$

where β_k ($\le \alpha_k$) is a fixed numerical constant. This result (B), for $k=2$, is relevant to the travelling-salesman problem discussed by Heller [Proc. 2nd Symp. Linear Programming, 1955, pp. 643–665, Nat. Bur. Standards, Washington, D.C., 1955; MR **17**, 873], Marks [Ann. Math. Statist. **19** (1948), 419–422; MR **10**, 131], Morton and Land [J. Roy. Statist. Soc. Ser. B **17** (1955), 185–194; MR **19**, 175]. More generally, if $P_1, P_2, \cdots$ are independently distributed over E with a common probability density function p, the density being taken with respect to Lebesgue measure, then with probability one,

$$\text{(C)}\qquad \lim_{n\to\infty} n^{-(k-1)/k} l(\mathbf{P}^n) = \beta_k k^{1/2} \int_E p^{(k-1)/k} dv.$$

Thus, when p is not uniform over E, the result (C) gives a shorter expected path length than (B).

The authors point out that, with suitable modifications, their results are relevant to Steiner's street network problem and to the Loberman-Weinberger wiring problem. They have possible generalisations in the direction of Plateau's problem and Douglas' problem.

S. J. Taylor (Ithaca, N.Y.)

DIFFERENTIAL GEOMETRY

See also 123, 230.

203:

Ionesco, Haralambie P. Une représentation vectorielle des courbes planes et quelques propriétés. Bul. Inst. Politehn. București **20** (1958), no. 2, 23–32. (Russian, English and German summaries)

204:

Jovanović, Božidar D. Quelques théorèmes sur les systèmes des forces dans espace. Univ. Beograd. Godišnjak Filozof. Fak. Novi Sad **2** (1957), 381–387. (Serbo-Croatian. French summary)

Dans la note présente on étudie les deux théorèmes suivant sur les vecteurs glissants: (1) la condition nécessaire de la coupure des axes centrales de deux torseurs; (2) le théorème de Morley-Petersen que trois droites spéciales coupent une même droite à angle droit. La preuve est donnée à l'aide de l'algèbre des torseurs.

D. Rašković (Belgrade)

205:

Georgiev, G. Sur la géométrie différentielle des champs de vecteurs et quelques applications. Bŭlgar. Akad. Nauk

Izv. Mat. Inst. **4**, no. 1, 107–120 (1959). (Russian. Bulgarian and French summaries)

Résumé de l'auteur: "On expose les principaux résultats obtenus récemment par l'auteur dans le domaine de la géométrie différentielle des champs de vecteurs unitaires et des complexes de droites, de même que les applications à l'étude de quelques problèmes d'hydrodynamique."

206:

Kecs, W. L'indicatrice de la torsion géodésique et de la courbure normale et quelques applications. Gaz. Mat. Fiz. Ser. A **10** (**63**) (1958), 646–651. (Romanian. French and Russian summaries)

Three theorems concerning certain properties of a curve and its conjugate are announced. The proof of the theorems is geometrical, based upon a graphical interpretation of the formulae expressing the normal curvature and the geodesic torsion of a curve on a surface.

G. Soós (Debrecen)

207:

Vogel, Walter O. Regelflächen im isotropen Raum. J. Reine Angew. Math. **202** (1959), 196–214.

Im isotropen Raum J_3 (d.h., im dreidimensionalen Raum der kartesischen Koordinaten x, y, z mit quadratischer einfach-singulärer Maßbestimmung $ds^2=dx^2+dy^2+0\cdot dz^2$) versteht der Verfasser unter einer Regelfläche eine einparametrige Schar von geraden Linien, die gewissen Stetigkeits und Differenzierbarkeitsbedingungen genügt und nicht in eine einzige Gerade ausgeartet ist.—Im euklidischen Raum R_3 kennt man drei Typen von Regelflächen (Zylinderflächen, Kegelflächen und allgemeine Regelflächen einschließlich Torsen). Demgegenüber gibt es in J_3 fünf Typen. Typus I: Regelflächen mit vollisotropen Erzeugenden. Typus II: die Grundrisse der Erzeugenden fallen in eine einzige Gerade zusammen. Typus III: die Grundrisse der Erzeugenden sind parallele Geraden. Typus IV: die Grundrisse der Erzeugenden bilden ein ebenes Strahlenbüschel mit eigentlichem Scheitel. Typus V: die Grundrisse der Erzeugenden sind Einhüllende einer ebenen Kurve.—Die zugehörigen Ableitungsgleichungen dieser Regelflächen führen im Falle I, II auf eine Krümmung, im Falle III, IV auf zwei Krümmungen und im Falle V auf drei Krümmungen. Sind die Krümmungen als Funktion eines invarianten Flächenparameters vorgegeben, so ist die zugehörige Regelfläche bis auf isotrope Bewegungen

$$\begin{aligned} x^1 &= a+x\cos\varphi-y\sin\varphi, \\ y^1 &= b+x\sin\varphi+y\cos\varphi, \\ z^1 &= c+c_1x+c_2y+z \end{aligned}$$

eindeutig bestimmt.—Weiterhin werden jene Regelflächen diskutiert, deren Krümmungen teilweise oder insgesamt konstant ausfallen bzw. verschwinden. Es gibt Minimalregelflächen und Analoga zu den geradlinigen uneigentlichen Affinsphären. Die Minimalregelflächen zerfallen in hyperbolische Paraboloide mit vollisotroper Durchmesserrichtung und isotrope Wendelflächen.—Die Cliffordschen Parallelismen in J_3 gestatten eine Abbildung der Flächenelemente des J_3 auf die Punktepaare einer Ebene (parataktische Abbildung). Dieser Zusammenhang wird für die parataktischen Links- und Rechtsbilder der berührenden Flächenelemente im Falle (regulärer) Regelflächen des isotropen Raumes von besonderem Interesse.

M. Pinl (Cologne)

208:

Delande, Georges. Sur une extension des surfaces minimales adjointes d'Ossian Bonnet. C. R. Acad. Sci. Paris **250** (1960), 49–51.

Author's summary: "Nous élaborons une théorie des champs géodésiques d'un espace fibré quadri-dimensionnel, et nous montrons qu'elle se trouve en relation étroite avec le problème variationnel de Koschmieder [Math. Z. **41** (1936), 43–55]. Nous traduisons ensuite, dans le langage des champs géodésiques, les propriétés classiques des surfaces minimales adjointes d'Ossian Bonnet."

T. K. Pan (Norman, Okla.)

209:

Osserman, Robert. An extension of certain results in function theory to a class of surfaces. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1031–1035.

The author considers a class $\mathfrak{S}(m)$ of differential-geometric surfaces: A surface S with the Riemannian metric ds^2 belongs to $\mathfrak{S}(m)$ if there exists a mapping of S into the complex z-plane which is conformal with respect to ds^2 and satisfies $ds\leq|dz|\leq m\,ds$ throughout S. A large number of results in function theory can be, in a generalized form, carried over to this class, e.g., the Koebe $\frac{1}{4}$-theorem. The class $\mathfrak{S}(m)$ is shown to include the class $\mathfrak{M}(\alpha)$ of all simply connected minimal surfaces S having the property that all normals to S make an angle of at least $\alpha>0$ with some fixed direction. As a result, a conjecture of L. Nirenberg, that every complete surface $S\in\mathfrak{M}(\alpha)$ is a plane, is proved.

K. Strebel (Fribourg)

210:

Barner, Martin; Degen, Wendelin. Über sich schliessende Figuren asymptotisch transformierter Kurven und W-transformierter Flächen. Math. Z. **71** (1959), 361–379.

Two curves $\mathfrak{p}(t)$ and $\mathfrak{q}(t)$ in real projective three-space are asymptotic transforms (of one another) if $(\mathfrak{q}\mathfrak{p}\dot{\mathfrak{p}}\ddot{\mathfrak{p}})=(\mathfrak{p}\mathfrak{q}\dot{\mathfrak{q}}\ddot{\mathfrak{q}})=0$ for all t, i.e., if they are asymptotic lines on the ruled surface which they generate. The authors construct systems of n curves $\mathfrak{p}_0(t), \mathfrak{p}_1(t), \cdots, \mathfrak{p}_{n-1}(t), \mathfrak{p}_n(t)=\mathfrak{p}_0(t)$ such that $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ are asymptotic transforms ($n\geq 4$; $i=0, 1, \cdots, n-1$). Two methods are developed. Let $\mathfrak{k}$ be a fixed non-degenerate linear complex. (i) All the lines $(\mathfrak{p}_i(t), \mathfrak{p}_{i+1}(t))$ belong to $\mathfrak{k}$. (ii) All the tangents of the $\mathfrak{p}_i(t)$ belong to $\mathfrak{k}$. The solutions of (i) and (ii) depend on at least four and $n-1$ arbitrary functions of one variable respectively. If $n=5$, all the solutions are obtained in this fashion.

"In enger Beziehung zu den asymptotisch transformierten Kurven stehen die W-transformierten Flächen: auf den beiden Brennflächen eines W-strahlsystems bilden entsprechende Asymptotenlinien sowohl der einen wie der andern Schar asymptotisch Transformierte voneinander ... [Mittels Kombination der beiden obigen Konstruktionen] entsteht ein sich schliessendes System von n-Flächen, dessen Verbindungskongruenzen W-strahlsysteme sind." The asymptotic lines of these surfaces are determined and the case $n=5$ is discussed in detail.

P. Scherk (Toronto, Ont.)

211:
Roman, A. Observations concernant la décomposition d'un complexe de droites dans l'espace euclidien. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) 4 (1958), 61–66. (Romanian. Russian and French summaries)

Applying Cartan's method of moving frames, the author attaches a special frame to a complex of lines in the three-dimensional euclidean space. The geometrical interpretation of some coefficients figuring in the Pfaffian forms of the moving frame is found. *G. Soós* (Debrecen)

212:
Hangan, T. Equations aux dérivées partielles satisfaites par les transformations du groupe projectif ou conforme. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 7 (1958), no. 19, 33–37. (Romanian. Russian and French summaries)

The author proves a theorem due to Vranceanu. The new proof differs only in technical aspects from the original proof of Vranceanu. *G. Soós* (Debrecen)

213:
Godeaux, Lucien. Sur l'envelope des quadriques de Lie d'une surface et sur une suite de quadriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) 45 (1959), 676–681.

Quatre points caractéristiques de la quadrique de Lie d'une surface se confondent au point de contact de la quadrique avec la surface; le quadrique de Lie a donc au plus cinq points caractéristiques. L'auteur a montré dans son célèbre mémoire sur *La théorie des surfaces et l'espace réglé* [Actualités Sci. Ind. No. 138, Hermann, Paris, 1934] que dans le cas où l'enveloppe de la quadrique de Lie comporte cinq nappes sur lesquelles sont conservées les asymptotiques, on peut associer à cette surface une suite de quadriques. Dans le présent article, l'auteur établit quelques propriétés de cette suite. *M. Decuyper* (Lille)

214:
Nijenhuis, Albert. A note on hyperconvexity in Riemannian manifolds. Canad. J. Math. 11 (1959), 576–582.

Let V be a geodesically convex open set of the Riemannian manifold M. V is called γ-hyperconvex if, for every $\varepsilon>0$ and any geodesic segments y_1z_1, y_2z_2 in V, the inequalities $\rho(y_1, y_2)<\varepsilon$, $\rho(z_1, z_2)<\gamma\varepsilon$ imply that corresponding (with respect to reduced arc-length) points of y_1z_1, y_2z_2 have a distance less than ε. Let D be the diameter of V, k the maximum of the curvatures of all 2-planes at points of V, and $m=\max\{k, 0\}$. Theorem: If $mD^2<\pi^2/4$, then V is γ-hyperconvex, where $\gamma=\cos Dm^{1/2}$. This implies that every point of a Riemannian manifold has a $\frac{1}{2}$-hyperconvex neighborhood, answering a question which arose in E. A. Michael's theory of continuous selections [see #230 below]. *L. W. Green* (Minneapolis, Minn.)

215:
Tachibana, Shun-ichi. On almost-analytic vectors in almost-Kählerian manifolds. Tôhoku Math. J. (2) 11 (1959), 247–265.

Let M be a differentiable manifold of even dimension which carries an almost complex structure given in local coordinates by the tensor $\phi^i{}_j$. Then by definition $\phi_r{}^i\phi_j{}^r=-\delta_j{}^i$. If it has a positive definite Riemannian metric g_{ji} satisfying $g_{rs}\phi_j{}^r\phi_i{}^s=g_{ji}$, then it is said to be almost-Hermitian. Finally, if the exterior differential form $\phi=\phi_{ji}dx^j\wedge dx^i$ is closed, then M is almost-Kählerian. This is a generalization of the notion of Kählerian manifold. For a different generalization, known as a pseudo-Kählerian manifold, it has been possible to generalize certain integral formulas and global theorems which hold in the case of Kählerian manifolds [K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1957; MR 19, 576]. In this paper the author gives many of the corresponding results for the almost-Kählerian case.

To give a sample of the results, we note that a Killing vector (field) is one relative to which the Lie derivative of g_{ji} vanishes, an (almost) analytic vector (field) is one relative to which the Lie derivative of $\phi_j{}^i$ vanishes, a harmonic vector (field) is defined as usual. From an integral formula which states that the integral over M of some function of a vector field v on M vanishes, it is possible to conclude—now always for compact M—that (1) an analytic vector on M is a Killing vector if and only if its covariant derivative vanishes; (2) if the curvature of g_{ji} is constant and negative there is no non-trivial analytic vector on M; and (3) if the metric M is an Einstein metric whose scalar curvature is negative, then there does not exist a non-trivial analytic vector.

W. M. Boothby (St. Louis, Mo.)

216:
Nagano, Tadashi. The projective transformation on a space with parallel Ricci tensor. Kōdai Math. Sem. Rep. 11 (1959), 131–138.

This paper proves two theorems on projectively related manifolds, as follows.

Theorem (1): Let g and $\hat{g}$ be complete Riemannian metrics on a connected manifold M whose Ricci tensors are parallel. If g and $\hat{g}$ are projectively related, then either their Levi-Civita connections coincide or they are both of constant positive curvature.

For the second theorem the following definitions must be made: if L and $\hat{L}$ are affine connections without torsion on M (as above), they are said to be projectively related if in local coordinates $\hat{L}_{jk}{}^i=L_{jk}{}^i+\delta_j{}^i\phi_k+\delta_k{}^i\phi_j$ for some vector field ϕ. Theorem (2): If L, $\hat{L}$ are projectively related, complete, torsion-free connections on M with parallel Ricci tensors, and if the (symmetrized) Ricci tensors are positive definite, then either L and $\hat{L}$ coincide or they are the Levi-Civita connections of Riemannian metrics of positive curvature. These results are related to a theorem of N. Tanaka [Nagoya Math. J. 12 (1957), 1–24; MR 21 #3899]. *W. M. Boothby* (St. Louis, Mo.)

217:
Dumitraş, V. La classification des espaces A_3 projectivement euclidiens. I. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. 7 (1958), no. 19, 19–32. (Romanian. Russian and French summaries)

The paper contains a complete classification of the three-dimensional projectively euclidean spaces having a transitive group of transformations. The classification is based upon the behaviour of the forms $f=P_{ij}dx^idx^j$ and $\varphi=\Gamma_{ji}dx^jdx^i$, the coefficients of which arise by contraction

from the curvature tensor: $P_{jk}=\Gamma^i_{ijk}$, $\Gamma_{jk}=\Gamma^i_{jki}$. Making use of convenient transformations of the coordinate system, the author characterizes the different space types by reducing the connection quantities to so-called canonical forms. Three theorems are proved, each referring to a case when one of the above-mentioned forms vanishes and the other has a rank different from zero.

G. Soós (Debrecen)

218:

Lelong-Ferrand, Jacqueline. Condition pour qu'un groupe de transformations infinitésimales engendre un groupe global. C. R. Acad. Sci. Paris **249** (1959), 1852–1854.

In her previous paper [J. Math. Pures Appl. (9) **37** (1958), 269–278; MR **20** #4874], the author studied the following problem: when does an infinitesimal transformation generate a 1-parameter group of global transformations? This paper generalizes her previous results. Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. A manifold V on which $\mathfrak{g}$ acts as a Lie algebra of infinitesimal transformations satisfies, by definition, the condition C if there exist a proper map $f: V\rightarrow G/H$ (where H is a closed subgroup of G) and a bounded map $A: V\rightarrow E(\mathfrak{g})$ (where $E(\mathfrak{g})$ is the set of endomorphisms of $\mathfrak{g}$) such that, for every $x\in V$ and $Y\in\mathfrak{g}$, the vector field $A(x)Y$ on G/H evaluated at $f(x)$ coincides with the image under f of the vector Y_x. (Every element of $\mathfrak{g}$ defines a vector field on G/H as well as on V.) The main theorem states that the action of $\mathfrak{g}$ on V is induced by that of G if and (under the assumption that G is simply connected) only if V is the union of submanifolds M each of which is invariant by $\mathfrak{g}$ and satisfies C. An application to a Riemannian manifold is given.

S. Kobayashi (Vancouver, B.C.)

219:

Goldberg, S. I. On characterizations of the Euclidean sphere. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 384–390.

Stressing the relations between differential geometry and elliptic differential equations, the author gives proofs for two well-known theorems, due to H. Hopf: A closed orientable analytic surface of genus zero and constant mean curvature is a sphere. The only closed analytic special Weingarten surfaces of genus zero are spheres. The latter follows from a more general statement (theorem 3) of the author.

J. C. C. Nitsche (Minneapolis, Minn.)

220:

Grove, V. G. On closed surfaces. Univ. Nac. Tucumán. Rev. Ser. A **12**, 11–25 (1959). (Spanish)

Let $S: X=X(u^1, u^2)$ be a closed strictly convex surface of class C^m with the Gaussian curvature K. Let $f_{\alpha\beta}$ be a symmetric tensor of class C' on S which satisfies $f_{\alpha\beta,\gamma}=f_{\alpha\gamma,\beta}$ such that the form $O=f_{\rho\sigma}du^\rho du^\sigma$ is definite. Put $J=|f_{\alpha\beta}|/g$, where g is the discriminant of the first fundamental form of S. Suppose another surface $\bar S$ and the tensor $\bar f_{\alpha\beta}$ on $\bar S$ satisfy the same assumptions. Denote the corresponding scalars by $\bar K$ and $\bar J$. If the continuously differentiable homeomorphism $h: S\rightarrow\bar S$ satisfies $O=\bar O$, $J=\bar J$, $K=\bar K$, then S and $\bar S$ are congruent. The proof consists in showing by means of an integral formula that h must be an isometry.

P. Scherk (Toronto)

GENERAL TOPOLOGY, POINT SET THEORY

221:

Mayrhofer, Karl. Begründung einer Topologie in Somenräumen. Monatsh. Math. **62** (1958), 277–296.

Carathéodory generalized measure theory by replacing sets of points by elements of an abstract Boolean algebra [*Mass und Integral und ihre Algebraisierung*, Birkhauser, Basel and Stuttgart, 1956; MR **18**, 117]. He called such an element a soma. In this article the author generalizes topology in the same way.

In what he and Carathéodory call a soma-space (Somenraum) and others would call a complete Boolean algebra, a topology is introduced by specifying which elements are to be considered open, subject to the usual requirements that finite "intersections" and arbitrary "unions" of open elements are open. In terms of this basic notion of open element, the usual notions of closed element, closure of an element, boundary, frontier, connected element, nowhere dense, compact and countably compact are defined in exactly the usual way, and are proved to have the usual properties. The usual consequences of the second countability axiom are derived. No separation axioms are considered, although the concept of a normal soma-space would seem to be a natural one. No references are given to similar investigations in the literature. It is pointed out that M. H. Stone's theorem on the representation of Boolean algebras by sets does not reduce the present theory to ordinary set topology, since this representation preserves only finite unions and intersections. A soma-space need not be "atomic".

O. Frink (University Park, Pa.)

222:

Helmberg, Gilbert. Uniform convergence of nets of functions. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 419–427; corrigendum, **63** (1960) [Indag. Math. **22** (1960)], 106.

Let S be a family of continuous functions on a space X. The author examines conditions under which properties of Dini type for S imply the compactness or local compactness of X. A typical theorem is the following. Let S be a family of non-negative functions on a set X, which contains with any of its members all its positive multiples and positive powers and with any pair the max and min. Let T be the weakest topology on X which turns S into a family of upper semi-continuous functions on X. Then T is compact if and only if the Dini property holds for S, i.e., any subfamily of S which is directed downwards and converges to zero converges uniformly. Various other related results are proved.

W. T. van Est (Leiden)

223:

Gillman, L.; Jerison, M. Stone-Čech compactification of a product. Arch. Math. **10** (1959), 443–446.

The authors formulate a variant of the problem of when $\beta(X\times Y)=\beta X\times\beta Y$; namely, when are these two spaces homeomorphic? They show that the answer is the same as for the original problem [I. Glicksberg, Trans. Amer. Math. Soc. **90** (1959), 369–382; MR **21** #4405] if both X and Y are first countable, but not always. More details are given for locally compact σ-compact spaces.

J. Isbell (Seattle, Wash.)

224:

Treybig, L. B. Concerning local separability in locally peripherally separable spaces. Proc. Amer. Math. Soc. **10** (1959), 957–958.

Suppose that S is a connected metric topological space which is not separable (e.g., uncountably many straight line rays emanating from a common point such that each two of them intersect in an angle of 180 degrees). No such space S is locally separable. However, S may be locally peripherally separable even when the set M of points at which S is not locally separable is itself separable. But it is shown in this paper that in any case where S is locally peripherally separable M must be uncountable.

F. B. Jones (Chapel Hill, N.C.)

225:

Fischer, H. R. Limesräume. Math. Ann. **137** (1959), 269–303.

The author develops in detail the theory of a general type of limit space, based on a determination of which filters of the space converge to which points. If we call a filter converging to x an x-filter, he assumes always that the intersection of two x-filters is an x-filter, that any filter including an x-filter is an x-filter, and the the ultrafilter of all sets with x as member is an x-filter. For a given x, the set of all x-filters is a dual ideal in the partially ordered set of all filters.

Topologies in the usual sense occur as special cases in which the x-filters are just those filters which contain all neighborhoods of x. A limit space always determines a particular topology, but is not determined by it.

In terms of this limit notion, the author defines and studies compactness, adherence, closure, separation axioms, and continuity of mappings. He investigates also quotient spaces, sum spaces, and product spaces, and proves a generalization of the Tychonov theorem that the product of compact spaces is compact.

He then generalizes the notions of topological group and topological linear space, to groups with limits and linear limit spaces. For a group with limits he defines completeness and Cauchy filter, but postpones to another article the study of the completion problem. Finally he defines the inductive limit of a family of limit spaces, and shows by examples that with the inductive limit, the property of being a topological space is not always preserved.

O. Frink (University Park, Pa.)

226:

Kim, Chi Young. On definitions of a uniform space by the convergence class. Kyungpook Math. J. **2** (1959), 1–6.

A convergence class is a triple (L, D, N), where L is a class of sequences of points in a space X, D is an abstract directed set, and N is a function on $L\times D$ to the natural numbers. The author gives necessary and sufficient conditions on (L, D, N) in order that L should be the class of all convergent sequences relative to a uniform structure on X. D plays the part of a base for the uniformity, and for $s\in L$, $d\in D$, $N(s, d)$ corresponds to the smallest n such that $s_k\in d[x]$ for all $k\geqq n$, where $x=\lim_n s_n$. The author neglects to attach the intended limits firmly to the sequences of L, i.e., L should be taken as a class of ordered pairs (s, x) where s is a sequence and x its limit. There is also some confusion arising from failure to distinguish between a sequence and its range.

E. Kallin (Berkeley, Calif.)

227:

Ball, B. J. Certain collections of arcs in E^3. Proc. Amer. Math. Soc. **10** (1959), 699–705.

Let G be a continuous collection of disjoint sets filling up a compact subset of E^3 and having a Cantor set as decomposition space; i.e., let G be the collection of point inverses under an open mapping of a compact subset of E^3 onto a Cantor set. Assume further that there are horizontal planes α and β such that G is a collection of arcs each having one end point on α and the other on β and none having two points on any horizontal plane. The author gives necessary and sufficient conditions for there to exist a homeomorphism of E^3 onto itself taking each element of G onto a vertical interval and not changing the ζ-coordinate of any point. He also gives an example of a collection G not satisfying these conditions in which the elements are straight line intervals.

E. Dyer (Chicago, Ill.)

228:

Kwun, Kyung Whan. A generalized manifold. Michigan Math. J. **6** (1959), 299–302.

The author constructs a sphere-like 3-gcm which is not locally euclidean at any point. His example is the decomposition space of a certain upper semicontinuous decomposition of a 3-sphere into points and wild arcs.

E. Dyer (Chicago, Ill.)

229:

Michael, E. A theorem on semi-continuous set-valued functions. Duke Math. J. **26** (1959), 647–651.

A mapping ϕ from a space X to the space 2^Y of non-empty subsets of a space Y is said to be "lower semi-continuous" (l.s.) if, for each open U in Y, the set $\{x|\ x\in X, \phi(x)\cap U\neq\emptyset\}$ is open in X; it is "upper semi-continuous" (u.s.) if $\{x|\ x\in X, \phi(x)\subset U\}$ is open. If $f:Y\rightarrow X$ is onto, it is open [closed] if and only if $f^{-1}:X\rightarrow 2^Y$ is l.s. [u.s.]. Theorem: If X is paracompact and Y is metric, and $\phi:X\rightarrow 2^Y$ is l.s. and such that each $\phi(x)$ is complete, then there exist l.s. ψ and u.s. θ from X to 2^Y such that, for each $x\in X$, $\psi(x)$ and $\theta(x)$ are compact and $\psi(x)\subset\theta(x)\subset\phi(x)$. This theorem is applied to give results about open mappings of spaces, including the following. Suppose E is a metric space and F is Hausdorff, and $f:E\rightarrow F$ is open and onto (but not necessarily continuous) and such that each set $f^{-1}(y)$ $(y\in F)$ is complete. Then, given any compact $B\subset F$, there exists a compact $A\subset E$ such that $f(A)=B$.

A. H. Stone (Manchester)

230:

Michael, Ernest. Convex structures and continuous selections. Canad. J. Math. **11** (1959), 556–575.

Let $\mathfrak{S}$ be a family of subsets of the space Y, ϕ a lower semicontinuous mapping of a paracompact space X into $\mathfrak{S}$. A selection for ϕ is a continuous function $f:X\rightarrow Y$ such that $f(x)\in\phi(x)$ for all x. The author continues his study of the conditions under which selections exist and are extendible [see the author, Ann. of Math. (2) **63** (1956), 361–382; **64** (1956), 562–580; **65** (1957), 375–390; MR **17**, 990; **18**, 325, 750]. The previously obtained affirmative answer for the existence of a selection for every such ϕ when Y is a Banach space and $\mathfrak{S}$ the family of non-empty, closed convex subsets of Y is generalized to the situation where Y is metric with "convex structure",

$\mathfrak{S}$ a family of non-empty, complete, "convex" subsets of Y. The definition of a convex structure on Y is too long to reproduce here; the general idea is that the existence of a distinguished family of singular simplexes is postulated which enables one to take "continuous convex combinations" of certain n-tuples of points, for all n. It is shown that a Riemannian manifold may be supplied with a (local) convex structure related naturally to the geodesics. [Cf. #214 above.] Applications are given to locally convex metrizable linear spaces and Lie groups (including Gleason's cross-section theorem, at least for metrizable groups). The concept of a uniformly equi-locally convex family of subsets of a metric space with convex structure is used to obtain a selection theorem which includes both a homotopy extension property and a covering homotopy theorem. This is applied to fibre spaces in the sense of Hurewicz. *L. W. Green* (Minneapolis, Minn.)

231:

Titus, C. J.; Young, G. S. An extension theorem for a class of differential operators. Michigan Math. J. 6 (1959), 195–204.

Étude des courbes de Loewner définies par une représentation de la forme

$$x = ((P_n(D))(f))(t),$$
$$y = ((P_{n-1}(D))(f))(t)$$

où P_n et P_{n-1} sont des polynômes (D opérateur d/dt) dont les racines vérifient certaines relations simples. Ces courbes ont un indice $\geqq 0$ en tout point du plan (courbes à circulation $\geqq 0$). Le résultat essentiel de ce mémoire établit que les courbes de Loewner sont les images de la frontière d'un cercle par une application $\varphi \in \phi$: où ϕ est la famille des applications continues sur le disque fermé, intérieures légères sur l'intérieur du cercle.

Un contre-exemple prouve que la nonnégativité de la circulation (même lorsque l'indice d'enroulement tangentiel est 1) ne caractérise pas les images d'une circonférence par les applications $\in \phi$. Toutefois un résultat partiel (théor. 2) est obtenu dans la recherche de cette caractérisation. *L. Fourès* (Marseille)

ALGEBRAIC TOPOLOGY

See also 228, 230.

232:

Gil de Lamadrid, J.; Jans, J. P. The set of all covering spaces. Proc. Amer. Math. Soc. 10 (1959), 710–715.

From the introduction: "Recently, Lee extended the definition of covering space to spaces which are not necessarily locally connected [Duke Math. J. 24 (1957), 547–554; MR 19, 668]. He showed that many of the classical results [C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, 1946; MR 7, 412; Chapters II, VI–X] on covering spaces and simple connectedness can be extended to this wider class of spaces. In this paper, using a modification of Lee's definition, we consider the structure of the set $C(X)$ of "all" covering spaces of a space X. We put an order on $C(X)$ and consider conditions under which $C(X)$ has a last element, a universal covering space. We make a distinction between simply connected covering space and universal covering space. Although simply connected covering spaces are universal, the converse is an open question. We also show that a universal covering space, if it exists, is unique. The last section gives examples." *D. Montgomery* (Princeton, N.J.)

233:

Fadell, Edward. On fiber homotopy equivalence. Duke Math. J. 26 (1959), 699–706.

A. Dold has given a criterion for two fiber bundles over a common polyhedral base with locally compact fibers to be fiber homotopy equivalent [Math. Z. 62 (1955), 111–136, MR 17, 519]. In this paper the author shows that Dold's criterion holds for Hurewicz fibrations with no local compactness assumption on the fibers involved.

As an application, it is proved that any universal bundle over a polyhedron P whose group is dominated by a CW-complex is fiber homotopy equivalent to the fiber space of paths emanating from a fixed point of P. *W. S. Massey* (Providence, R.I.)

234:

Tominaga, Akira. A condition under which simple closed curves bound discs. J. Sci. Hiroshima Univ. Ser. A 22, 205–214 (1958).

Let M be a 3-manifold, compact or not, with boundary and let L be a simple closed curve in M. If for each pair of distinct points $a, b \in L$ and curve C joining a and b in $M-L$ there exists a curve C' in L joining a and b and homotopic to C and, moreover, such that the image of the interior of the square of the homotopic mapping is in $M-L$, then one writes $\pi_1(M-L, L)=1$. It is shown that a necessary and sufficient condition that L bounds a disc in M is that there exists a neighborhood U of L such that L is homotopic to zero in U and $\pi_1(U-L, L)=1$. *W. R. Utz* (Columbia, Mo.)

235:

Ringel, Gerhard. ★Färbungsprobleme auf Flächen und Graphen. Mathematische Monographien, 2. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. viii+132 pp.

In this book the author presents the results of his researches on the chromatic numbers of the closed surfaces. For surfaces other than the sphere he finds that Heawood's upper bound exceeds the true chromatic number by at most 2. Moreover Heawood's formula gives the chromatic number exactly for all non-orientable surfaces except the Klein bottle. For all surfaces, with the possible exception of the sphere, the chromatic number χ is equal to the maximum number ν of mutually adjacent regions which can occur in a map on the surface.

The first chapter is an introduction to the theory of graphs and the second is a very clear account of the Four Colour Problem. The latter is especially valuable for its careful proofs of the equivalence of the Four Colour Conjecture to the conjecture of Tait on the 3-colourability of the edges of a planar cubic graph and to the conjecture of Heawood on the solvability of a system of congruences modulo 3 associated with the vertices.

In Chapter 3 we are introduced to the problem of colouring a graph G so that no two vertices of the same colour are joined by an edge. The least number of colours for which this is possible is the chromatic number $\chi(G)$

of G. The graph is called critical if every proper subgraph has a smaller chromatic number. A few simple theorems about critical graphs are proved. They are then used in a proof of the 12-colour theorem for maps on the sphere in which each country consists of two connected pieces.

This chapter contains the proof, hitherto (I believe) unpublished, of a remarkable theorem of Hajós stating that any finite graph of chromatic number k can be derived from one or more complete $(k+1)$-graphs, in a finite number of steps, by two very simple operations. Attention is called to Hajós' Conjecture, a generalization of the Four Colour Conjecture. This asserts that every finite graph of chromatic number k contains a subdivision of a complete $(k+1)$-graph.

Chapters 4 and 5 are concerned with the decomposition of a regular graph into regular factors. The discussion follows closely that given in the text-book of D. König.

In Chapter 6 the author defines a closed surface in terms of a set of polygons whose edges are to be identified in pairs, and gives an excellent account of the reduction of such a surface to normal form by means of two operations of subdivision and their inverses. The surfaces are classified as the sphere, the orientable surfaces of Euler characteristic $2-2p$, and the non-orientable surfaces of characteristic $2-q$. The "genus" p or q takes all positive integral values. Models are given for these surfaces. There follows a discussion of bounded surfaces which is open to criticism. It is asserted without justification, combinatorial or otherwise, that a given closed arc on a surface is an edge in some polygonal dissection. Further the discussion is restricted to surfaces for which the boundary consists of disjoint simple closed curves, but in later chapters the results are applied to bounded surfaces which are not necessarily of this form (e.g., Chapter 7, Theorem 4).

In Chapter 7 the author establishes Heawood's formula

$$\nu \leqq \chi \leqq [\tfrac{1}{2}(7+(49-24N)^{1/2})],$$

where N is the Euler characteristic of the surface concerned. In Chapter 8 the cases $p=1$ and $q=1, 2, 3$ and 4 are analysed. For $q=2$ we have

$$6 = \nu = \chi < \tfrac{1}{2}(7+(49-24N)^{1/2}) = 7,$$

but in the other cases both equalities are valid. From later references it appears that a discussion of the cases $p=2$ and $p=3$ has been inadvertently omitted. Some matrices representing maps on higher surfaces are given. Using these, and some auxiliary results relating values of ν for different surfaces, it is possible to verify the Heawood equalities for $5 \leqq q \leqq 13$ and $4 \leqq p \leqq 9$.

In Chapter 9 the author embarks on his general theory. He gives a construction for a matrix S_{3n} of order $3n$, where n is any odd integer $\geqq 3$. He shows that this matrix represents a subdivision of a non-orientable surface into $3n$ mutually adjacent polygons, each with $3n-1$ edges. Starting with these surfaces, and using a simple construction for increasing q, he is able to show that

$$[\tfrac{1}{2}(7+(1+24q)^{1/2})]-2 \leqq \nu \leqq \chi \leqq [\tfrac{1}{2}(7+(1+24q)^{1/2})]$$

for all non-orientable surfaces.

An analogous but more difficult theory of the orientable surfaces is given in Chapter 10. It leads to the corresponding formula

$$[\tfrac{1}{2}(7+(1+48p)^{1/2})]-2 \leqq \nu \leqq \chi \leqq [\tfrac{1}{2}(7+(1+48p)^{1/2})].$$

In Chapter 11 the equality of ν and χ is established for all non-spherical surfaces. There remains the problem of determining whether the second equality in the Heawood formula holds for all surfaces other than the sphere and the Klein bottle. In Chapter 12 it is shown that the equality does hold for the non-orientable surfaces. The proof requires a much deeper theory of representative matrices than that used in the preceding chapters. The problem for orientable surfaces remains unsolved.

The reviewer recommends this book to the attention of mathematicians both on account of the excellent expository material in the first eight chapters and for the careful and successful analysis of the general problem in the remainder of the book. *W. T. Tutte* (Toronto)

DIFFERENTIAL TOPOLOGY

See also 215, 218.

236:

James, I. M. Some embeddings of projective spaces. Proc. Cambridge Philos. Soc. **55** (1959), 294–298.

Let $P_n(R)$, $P_n(C)$, and $P_n(Q)$ denote n-dimensional real, complex, and quaternionic projective spaces respectively. The main results of this paper are the following. Theorem 1: For $n \geqq 1$, $P_n(R)$ can be embedded in Euclidean $2n$-space. If n is odd and $\geqq 3$, then $P_n(R)$ can be embedded in $(2n-1)$-space. Theorem 2: Let $n \geqq 1$; then $P_n(C)$ can be embedded in $(4n-1)$-space. Moreover, if n is odd and $\geqq 3$, then $P_n(C)$ can be embedded in $(4n-3)$-space. Theorem 3: For $n \geqq 1$, $P_n(Q)$ can be embedded in $(8n-3)$-space. Theorem 4: The Cayley projective plane can be embedded in 25-dimensional Euclidean space.

All these embeddings are constructed explicitly, and the author shows that they are differentiable and regular. Theorem 1 had been proved previously by H. Hopf [Vierteljschr. Naturf. Ges. Zürich **85** (1940), Beiblatt, 165–177; MR **2**, 321] but without consideration of regularity. In most cases the author does not discuss whether or not these embeddings are in the lowest possible dimensional Euclidean space.

W. S. Massey (Providence, R.I.)

237:

Massey, W. S. On the normal bundle of a sphere imbedded in Euclidean space. Proc. Amer. Math. Soc. **10** (1959), 959–964.

The main theorem proved in this note states that: if M^n is a compact, differentiable, orientable manifold with the same integral homology groups as an n-sphere, then the normal bundle to M^n is of the same fibre homotopy type as a product bundle for any imbedding in Euclidean space of any dimension. As a corollary of this theorem, the author also proves that, for any imbedding of a differentiable manifold which is homeomorphic to an n-sphere in $(n+3)$-dimensional Euclidean space, the normal bundle to M^n is trivial. By using this corollary and some known results, one sees that for $n \leqq 6$, any imbedding of an n-sphere in Euclidean space of any dimension gives rise to a trivial normal bundle. It is an open question whether or not it is possible to imbed a 7-sphere in Euclidean 11-space with a nontrivial normal bundle. *Sze-Tsen Hu* (Los Angeles, Calif.)

238:

Vesentini, Edoardo. Sopra i sistemi fibrati kähleriani compatti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 657–661.

Let V and B be compact, connected, respectively n- and b-dimensional Kähler manifolds ($0 \leq b \leq n$). A map $\pi : V \to B$ is called a holomorphic projection if it is everywhere a holomorphic map, and if there is a complex subvariety $L \subset B$, $L \neq B$ (possibly with singularities or empty), such that for each $\xi \in B - L$ the subvariety $M_\xi = \pi^{-1}(\xi)$ is a nonsingular, connected, $(n-b)$-dimensional, complex analytic submanifold of V, and if, everywhere in $V - \pi^{-1}(L)$, the functional rank of π is b. Theorem 1: Let $\pi : V \to B$ be a holomorphic projection of a compact, Kählerian V onto a nonsingular, irreducible algebraic curve B (i.e., $b=1$). If the geometric genus $g_n(V)$ of V vanishes and the Riemannian genus $g(B)$ of B is $\geqq 2$, then the fibres $M_\xi = \pi^{-1}(\xi)$ for $\xi \in B - L$ have geometric genus $g_{n-1}(M_\xi)$ equal to zero. Theorem 2: If B is a complex projective b-space and $\pi : V \to B$ a holomorphic projection with L empty, then any holomorphic exterior form ω on V, whose restriction to M_ξ vanishes for each $\xi \in B$, is identically zero. Corollary 3: Under the hypotheses of Theorem 2, the genera $g_q(V)$ (i.e., the dimensionalities of the spaces of holomorphic q-forms on V) vanish for $n-b<q \leq n$. The above results are compared with earlier, less general ones, by F. Enriques ($n=2$, $b=1$) and by A. Borel (applicable when π is the projection map of a complex analytic fibre bundle).

E. Calabi (Minneapolis, Minn.)

239:

Morimoto, Akihiko. Sur la classification des espaces fibrés vectoriels holomorphes sur un tore complexe admettant des connexions holomorphes. Nagoya Math. J. **15** (1959), 83–154.

The author first proves the following theorem. A holomorphic vector bundle E over a complex torus $T^n = C^n/\pi$ which has a holomorphic connection has necessarily an integrable holomorphic connection, and hence is associated to a representation φ of the fundamental group π. If moreover E is indecomposable then, according to Matsushima [same J. **14** (1959), 1–24; MR **21** #1403], we can take $\varphi = \sigma \otimes \rho$ where σ is a 1-dimensional representation (uniquely determined by E) and ρ is a unipotent representation. Since the 1-dimensional representations give rise just to the Picard variety of T^n the problem is essentially reduced to the classification of unipotent representations of π with respect to the equivalence: $\rho \sim \rho'$ if the associated vector bundles E_ρ, $E_{\rho'}$ are isomorphic.

As a first step in this classification the author shows that the vector bundle E_ρ has a trivial sub-bundle E_ρ^0 generated by the global cross-sections of E_ρ, and that $E_1 = E_\rho/E_\rho^0$ has again a holomorphic connection. Repeating this procedure one defines $E_i = E_{i-1}/E_{i-1}^0$ and obtains a sequence of vector bundles

$$E_\rho = E_0, E_1, E_2, \cdots, E_{l-1} \neq 0, \quad E_l = 0$$

canonically determined by E_ρ. The sequence of integers $(m_1, \cdots, m_l)$, $m_i = \dim \Gamma(E_{i-1}) = \dim E_{i-1}^0$, is called the type of E_ρ. The set of isomorphism classes of indecomposable vector bundles associated to unipotent representations and having type $(m_1, \cdots, m_l)$ is denoted by $J(T^n; m_1, \cdots, m_l)$.

The second half of the paper is devoted to the explicit determination of the sets $J(T^n; m_1, \cdots, m_l)$ for $m = \sum m_i \leq 4$. The case $m=1$ is trivial and $m=2$ has been dealt with by Matsushima [loc. cit.]. For example, $J(T^n; 1, 2)$ turns out to be the Grassmannian of C^2's in C^n (a similar result holds also with 2 replaced by m).

M. F. Atiyah (Cambridge, England)

[illegible]

AUTHOR INDEX

PART A